MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A

MRC Technical Summary Report #2527

AN ASYMPTOTIC ANALYSIS OF
SINGLE-JUNCTION SEMICONDUCTOR DEVICES

Peter A. Markowich,
C. A. Ringhofer,
E. Langer and S. Selberherr

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

June 1983

(Received November 9, 1982)

Approved for public release
Distribution unlimited

DTIC
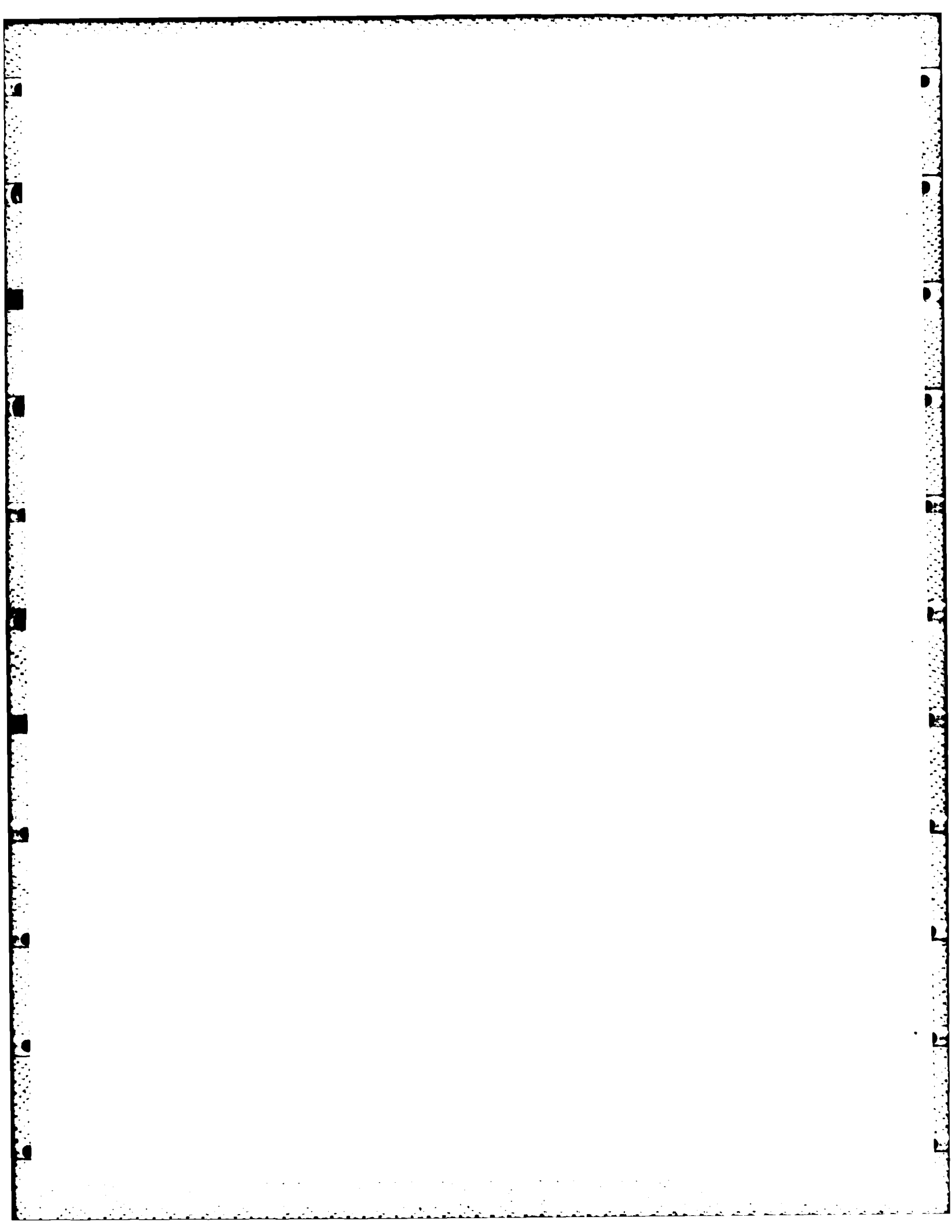ELECTE
JUL 2 1 1983
E

83 07 20 041

SIGNIFICANCE AND EXPLANATION

We present an analysis of a one-dimensional model of single-junction semiconductor devices (pn-junctions and certain resistors) when an external voltage is applied to the contacts. The model has the form of a system of six highly nonlinear first order ordinary differential equations subject to boundary conditions at the contacts of the device. The system is singularly perturbed (the derivatives of some components are multiplied by a small constant, the so called singular perturbation parameter). The dependent variables are the electrostatic potential, the hole and electron densities and the hole and electron current densities. A region of fast variation in the electrostatic potential and in the carrier distributions occurs due to the singular perturbation character of the problem. This region is in the interior of the device (internal layer) and represents the junction between differently doped areas. We derive formal asymptotic expansions of solutions as the singular perturbation parameter tends to zero and we prove that such an expansion 'represents' a solution. We also investigate the dependence of the total current on the externally applied voltage (voltage-current characteristic).

Accession For	
NTIS GRA&I	☒
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A	

MATHEMATICS RESEARCH CENTER

AN ASYMPTOTIC ANALYSIS OF SINGLE-JUNCTION SEMICONDUCTOR DEVICES

Peter A. Markowich[*], C. A. Ringhofer[**],
E. Langer[***] and S. Selberherr[***]

ABSTRACT

In this paper we present an analysis of the fundamental one-dimensional semiconductor equations describing potential, carrier, and current density distributions in single-junction semiconductor devices when an external voltage is applied to the contacts. We reformulate the model equations by appropriate scaling as a singularly perturbed two point boundary value problem for a system of nonlinear ordinary differential equations. The right-hand side of the system has a jump discontinuity with respect to the independent variable (space-coordinate) representing the junction between differently doped sides of the device. The solution components are assumed to be continuous across this junction.

We give an existence proof for the reduced problem (the singular perturbation parameter is set to zero). The discontinuity of the right-hand side of the system produces a discontinuity in the reduced potential and reduced carrier distributions. This creates an internal layer in the corresponding solution components of the singularly perturbed problem. The current distributions have no internal layer. We also derive the (internal) layer equations and give an existence proof. No boundary layers occur.

We show that formal expansions actually represent (asymptotically) solutions of the singularly perturbed problem if the applied voltage is sufficiently small, and we investigate the dependence of the total current on the applied voltage. Numerical computations are reported.

AMS (MOS) Subject Classifications: 34C11, 34D15, 34E15

Key Words: semiconductor devices, singularly perturbed ordinary differential equations, asymptotic expansions, internal layers.

Work Unit Number 1 - Applied Analysis

[*]
Institut fuer Angewandte und Numerische Mathematik, Technische Universität Wien, A-1040 WIEN, AUSTRIA.
[**]
Mathematics Research Center, University of Wisconsin-Madison, Madison, WI 53705.
[***]
Institut fuer Allgemeine Elektrotechnik, Technische Universität Wien, A-1040 WIEN, AUSTRIA.

AN ASYMPTOTIC ANALYSIS OF SINGLE-JUNCTION
SEMICONDUCTOR DEVICES

Peter A. Markowich[*], C. A. Ringhofer[**],
E. Langer[***] and S. Selberherr[***]

1. Introduction

In this paper we present an analysis of a class of systems of ordinary differential equations, subject to boundary conditions, modelling pn-junction devices. The physical situation is as follows. A semiconductor (for example Silicon) is doped with acceptor atoms (negative ions) in the left side, with donor atoms (positive ions) in the right hand side and a bias $U = U_A - U_C$ is applied to the contacts:

anode contact ——— | p – side | n – side | ——— cathode contact

U_A : Applied
anode potential

U_C : Applied
Cathode potential

−ℓ   0   Z   ℓ   z
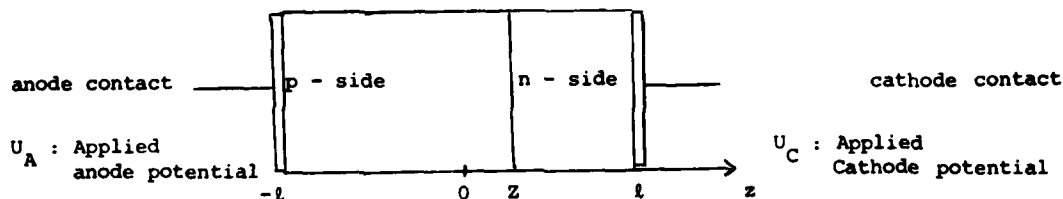
The device is assumed to have characteristic length $2\ell(\approx 5 \times 10^{-3} \text{cm})$ and the junction is at $z = Z \in (-\ell, \ell)$ (the term junction refers to the boundary of the n and p regions as well as to the whole device). The device is forward biased for $U > 0$ and reverse biased for $U < 0$. The physics of pn-junction is explained in Sze (1969), Ashcroft and (1976) and R. A. Smith (1978).

[*]
  Institut fuer Angewandte und Numerische Mathematik, Technische Universität Wien, A-1040 WIEN, AUSTRIA.
[**]
  Mathematics Research Center, University of Wisconsin-Madison, Madison, WI 53705.
[***]
  Institut fuer Allgemeine Elektrotechnik, Technische Universität Wien, A-1040 WIEN, AUSTRIA.

The equations describing the electrostatic potential, the carrier densities and the current densities within the device (in the static, one-dimensional case) are:

$$\text{(a)} \quad \psi'' = \frac{q}{\epsilon}(n - p - (N_D^+(z) - N_A^-(z))) \quad \text{Poisson's equation}$$

$$\text{(b)} \quad n' = \frac{\mu_n}{D_n} n\psi' + \frac{1}{qD_n} J_n \quad \text{electron current relation}$$

(1.1)
$$\text{(c)} \quad p' = \frac{\mu_p}{D_p} p\psi' - \frac{1}{qD_p} J_p \quad \text{hole current relation}$$

$$\text{(d)} \quad J_n' = qR(n,p,J_n,J_p) \quad \text{continuity equation for electrons}$$

$$\text{(e)} \quad J_p' = qR(n,p,J_n,J_p) \quad \text{continuity equation for holes}$$

for $z \in [-\ell,\ell]$ ("'" denotes differentiation with respect to z) subject to the boundary conditions

$$\text{(a)} \quad \psi(-\ell) = U_T \ell n \frac{n_i}{p(-1)} + U_A \quad \text{(anode)}$$

$$\text{(b)} \quad \psi(\ell) = U_T \ell n \frac{n(1)}{n_i} + U_C \quad \text{(cathode)}$$

(1.2)
$$\text{(c)} \quad n(\pm\ell)p(\pm\ell) = n_i^2$$

$$\text{(d)} \quad n(\pm\ell) - p(\pm\ell) = N_D^+(\pm\ell) - N_A^-(\pm\ell) \quad .$$

(See Van Roosbroeck (1950).)

The dependent variables (with units) in (1.1), (1.2) are

ψ : electrostatic potential (V)

ψ' : electrostatic field $(V cm^{-1})$

n : electron density (cm^{-2})

p : hole density (cm^{-3})

J_n : electron current density (A/cm^2)

J_p : hole current density (A/cm^2) .

All parameters in (1.1), (1.2) (except $N_A^-(z)$, $N_D^+(z)$) and the temperature T are assumed

to be constant. Table 1 gives the physical meaning and approximate numerical values of these parameters at $T \approx 300K$ (room temperature) for silicon.

Table 1: Parameters for Silicon at $T \approx 300K$

Parameter	Physical Meaning	Numerical Value
q	elementary charge	10^{-19}As
ϵ	permittivity constant	10^{-12}As/Vcm
μ_n	electron mobility	$10^3 cm^2$/Vs
μ_p	hole mobility	$10^3 cm^2$/Vs
D_n	electron diffusion constant	25 cm^2/s
D_p	hole diffusion constant	25 cm^2/s
n_i	intrinsic number	$10^{10} cm^{-3}$
$U_T = \dfrac{D_n}{\mu_n} = \dfrac{D_p}{\mu_p}$	thermal voltage	0.025V

N_A^- is the density of electrically active acceptor atoms and N_D^+ is the density of electrically active donor atoms and

(1.3) $C(z) = N_D^+(z) - N_A^-(z) \quad (cm^{-3})$

is called doping (or impurity) profile. For the pn-junction $C(z)$ is negative for $z \in [-\ell, Z)$ (p-side) and positive for $z \in (Z, \ell]$ (n-side) and is assumed to have a jump-discontinuity at $z = Z$ (abrupt junction). We also investigate the less important and much simpler case $C(z) > 0$ in $[-\ell, \ell]$ (but still with a jump-discontinuity at $z = Z$). These devices are called n^+n- or nn^+-junctions (depending on whether $C(z_1) > C(z_2)$ or $C(z_1) < C(z_2)$ for all $z_1 \in [-\ell, X]$, $z_2 \in (X, \ell]$).

The analysis of p^+p and pp^+ junctions ($C(z) < 0$ on $[-\ell, \ell]$) is analogous to the analysis of n^+n and nn^+ junctions. Only n and p, J_n and J_p have to be interchanged and ψ has to be substituted by $-\psi$.

The scalar function $R \in C([0,\infty)^2 \times R^2)$ in (1.1)(e) is called recombination term, it is the rate at which electron-hole carrier pairs are generated ($R < 0$) or recombine

(vanish) $(R > 0)$, $R(n,p,0,0) = 0$ for n, p such that $p = n_i^2$ holds (equilibrium condition).

The Shockley-Read-Hall (SRH) recombination term

(1.4) $$R = R_{SRH}(n,p) = \frac{np - n_i^2}{\tau_p(n+n_i) + \tau_n(p+n_i)} \quad (cm^{-3}s^{-1})$$

describing thermal recombination, where $\tau_n, \tau_p (\approx 10^{-6}s)$ are the electron and hole lifetimes, is widely used. Different ways of modelling R (which are necessary for very large $|U|$) are given in Langer, Selberherr and Mader (1981) and Schütz, Selberherr and Pötzl (1982).

The boundary conditions (1.2)(c) express that the contacts $z = \pm \ell$ are in thermal equilibrium and (1.2)(d) represents vanishing space charge at the contacts.

We only admit solutions of (1.1), (1.2) which fulfill

(1.5)(a) $$\psi, n, p, J_n, J_p \in c^1([-\ell,\ell])$$

(1.5)(b) $$\psi' \in c^1([-\ell,z]), \quad \psi' \in c^1([z,\ell])$$

(1.6) $$n \geqslant 0, \quad p \geqslant 0 \quad \text{on} \quad [-\ell,\ell] \ .$$

(1.5) comes from the jump-discontinuity of $C(z)$ (ψ'' cannot be continuous if n, p are continuous), the equation (1.1)(a) has to be fulfilled for the right hand limit and for the left hand limit of ψ'' at $z = Z$. (1.6) has to hold on physical grounds since n, p are densitites.

In this paper we scale (1.1), (1.2) such that we obtain a singular perturbation problem. The perturbation parameter (called λ in the sequel) is equal to $\frac{\lambda_D}{\ell}$ where $\lambda_D = \left(\frac{\varepsilon U_T}{q \max_{z \in [-\ell,\ell]} |C(z)|}\right)^{1/2}$ is the minimal Debeye length.

We present an asymptotic analysis of (1.1), (1.2) (for λ small, which corresponds to large doping $|C|$). The discontinuity of $C(z)$ at $z = Z$ produces an (internal) layer in the fast components ψ, ψ', n, p. J_n, J_p are the slow components (uniformly c^1 as $\lambda \to 0+$).

-4-

We derive the reduced problem $(\lambda = 0)$, the layer equations and give existence theorems for both. The reduced problem has the form of a two-point boundary value problem with interface conditions at the discontinuity. Using these results we prove an existence result for the full problem (1.1), (1.2) (for sufficiently large doping $|C|$, that means λ small) assuming that the recombination rate $R \equiv 0$ (corresponding to infinite electron and hole lifetime) and that $|U|$ is small. We show that (for λ sufficiently small) there is a solution of (1.1), (1.2) whose fast components are close to the sum of the (corresponding) 'reduced' solution components and the layer terms and whose slow components are close to the corresponding 'reduced' solution components. No layers at the contacts occur since the 'reduced' solution fulfills the 'reduced' boundary conditions.

We also investigate the dependence of the total current $J = J_n + J_p$ on the applied voltage U (J is a constant because of (1.1)(d),(e)1). It turns out that J is asymptotically (as $\lambda \to 0+$) a linear function of U if $C(z) > 0$ on $[-\ell, \ell]$ (n^+n and nn^+ junctions are resistors) and J is asymptotically an exponential function of U if C changes sign at z.

The singular perturbation approach to pn-junction modelling was suggested by many authors. Vasilev'a and Butuzow (1978), Vasilev'a and Stelmakh (1977) and D. Smith (1980) investigated a much simplified model (they assume that the current densities are known instead of the applied voltage, that $Z = 0$ and that $C(z)$ is odd) and prove an existence theorem using the asymptotic expansions. The authors of this paper analyzed (1.1), (1.2) (1982) under the (pretty unrealistic) assumption that the junction Z is in the middle of the device and that the doping profile C is odd. This allows to reduce the internal layer problem to a boundary layer problem. The advantages of the singular perturbation approach for the numerical solution of (1.1), (1.2) is also explained in the latter paper.

The generalization of the presented theory to multilayer structures like bulk-barrier diodes (see Langer, Selberherr and Mader (1981)) or thyristors (see Sze (1969)) is straightforward.

This paper is organized as follows. In Section 2 we perform the scaling and reformulation of (1.1), (1.2) as a singular perturbation problem, in Section 3 we derive the expansions, prove existence theorems for the reduced problem and the internal layer problem and in Section 4 we give the existence proof for $R \equiv 0$ the full singularly perturbed problem in the case λ and $|U|$ small. Numerical results for large U are demonstrated in Section 5.

2. Scaling

We scale the dependent variables as follows

$$\text{(2.1)} \qquad \psi_s = \frac{\psi}{U_T}, \quad n_s = \frac{n}{\bar{C}}, \quad p_s = \frac{p}{\bar{C}}$$

$$\text{(2.2)} \qquad J_{n_s} = \frac{\ell J_n}{D_n q \bar{C}}, \quad J_{p_s} = \frac{\ell J_p}{D_p q \bar{C}}$$

where $\bar{C} = \max\limits_{z \in [-\ell, \ell]} |C(z)|$ and the independent variable

$$\text{(2.3)} \qquad x = \frac{z}{\ell} \, .$$

Then (1.1) reads

(a) $\lambda^2 \psi_s'' = n_s - p_s - D(x)$

(b) $n_s' = n_s \psi_s' + J_{n_s}$

(c) $p_s' = -p_s \psi_s' - J_{p_s}$

(2.4)

(d) $J_{n_s}' = \dfrac{\ell^2}{D_n \bar{C}} \, R\!\left(\bar{C} n_s, \, \bar{C} p_s, \, \dfrac{D_n q \bar{C}}{\ell} J_{n_s}, \, \dfrac{D_p q \bar{C}}{\ell} J_{p_s}\right)$

(e) $J_{p_s}' = -\dfrac{\ell^2}{D_p \bar{C}} \, R\!\left(\bar{C} n_s, \bar{C} p_s, \dfrac{D_p q \bar{C}}{\ell} J_{n_s}, \dfrac{D_p q \bar{C}}{\ell} J_{p_s}\right)$

for $-1 < x < 1$ ("'" denotes now differentiation with respect to x). We have set

$$\text{(2.5)} \qquad D(x) = \frac{C(\ell x)}{\bar{C}}, \quad (|D(x)| \leq 1)$$

and

$$\text{(2.6)} \qquad \lambda^2 = \frac{\lambda_D^2}{\ell^2} = \frac{\varepsilon U_T}{\ell^2 q \bar{C}} \, .$$

The boundary conditions (2.2) are

(a) $\psi_s(-1) = \ell n\!\left(\dfrac{\gamma^2 \lambda^2}{p_s(-1)}\right) + \dfrac{U_A}{U_T}$

(b) $\psi_s(1) = \ell n\!\left(\dfrac{n_s(1)}{\gamma^2 \lambda^2}\right) + \dfrac{U_C}{U_T}$

(2.8)

(c) $n_s(\pm 1) p_s(\pm 1) = \gamma^4 \lambda^4$

(d) $n_s(\pm 1) - p_s(\pm 1) = D(\pm 1)$

-7-

where

(2.9)
$$\gamma^2 = \frac{\ell q n_i}{\varepsilon U_T}$$

holds.

We now assume that the recombination term R is such that

(2.10)
$$R\left(\overline{C}n_s, \overline{C}p_s, \frac{D_n q\overline{C}}{\ell} J_{n_s}, \frac{D_p q\overline{C}}{\ell} J_{p_s}\right) = \overline{C}S(n_s, p_s, J_{n_s}, J_{p_s}, \gamma\lambda)$$

holds, where $S \in C([0,\infty)^2 \times R^2 \times [0,\infty))$ is independent of $\overline{C}$.

Dropping the index s the problem now reads

(2.11)
$$\left.\begin{array}{ll} \text{(a)} & \lambda^2 \psi'' = n - p - D(x) \\[4pt] \text{(b)} & n' = n\psi' + J_n \\[4pt] \text{(c)} & p' = -p\psi' - J_p \\[4pt] \text{(d)} & J_n' = S_n(n, p, J_n, J_p, \gamma\lambda) \\[4pt] \text{(e)} & {}_p' = -S_p(n, p, J_n, J_p, \gamma\lambda) \end{array}\right\} \quad -1 < x < 1$$

with $S_n = \dfrac{\ell^2}{D_n} S$, $S_p = \dfrac{\ell^2}{D_p} S$ subject to the boundary conditions

(2.12)
$$\text{(a)} \quad \psi(-1) = \ell n\left(\frac{\gamma^2\lambda^2}{p(-1)}\right) + \frac{U_A}{U_T}$$

$$\text{(b)} \quad \psi(1) = \ell n\left(\frac{n(1)}{\gamma^2\lambda^2}\right) + \frac{U_C}{U_T}$$

$$\text{(c)} \quad n(\pm 1)p(\pm 1) = \gamma^4\lambda^4$$

$$\text{(d)} \quad n(\pm 1) - p(\pm 1) = D(\pm 1) \quad .$$

If $D_n = D_p = D$ holds we have

(2.13)
$$S_n = S_p \quad .$$

Under this assumption and $\tau_n = \tau_p = \tau$ we get for the SRH term with $\beta = \dfrac{D\tau}{\ell^2}$

(2.14)
$$S_n = S_p = \frac{1}{\beta} \frac{np - \gamma^4\lambda^4}{n + p + 2\gamma^2\lambda^2} \quad .$$

Generally, the equilibrium condition implies that $S_n(n, p, 0, 0, \gamma\lambda) = S_p(n, p, 0, 0, \gamma\lambda) = 0$ holds for n, p such that $np = \gamma^4\lambda^4$.

-8-

The discontinuity of D occurs at

(2.15) $$X = \frac{z}{\ell}$$

and the conditions on the solution of (2.11), (2.12) are

(2.16)   (a) $\psi, n, p, J_n, J_p \in C^1([-1,1])$

(2.16)   (b) $\psi' \in C^1([-1,X]), \ \psi' \in C^1([X,1])$

(2.17) $n > 0, \ p > 0 \quad \text{on} \quad [-1,1]$

(see (1.5), (1.6)).

The boundary values for n, p and ψ can be computed from (2.12)(c),(d):

(2.18) $n(1) = \frac{1}{2}\left(D(1) + \sqrt{D(1)^2 + 4\gamma^4\lambda^4}\right), \ p(1) = \frac{1}{2}\left(-D(1) + \sqrt{D(1)^2 + 4\gamma^4\lambda^4}\right)$

(2.19) $n(-1) = \frac{1}{2}\left(D(-1) + \sqrt{D(-1)^2 + 4\gamma^4\lambda^4}\right), \ p(-1) = \frac{1}{2}\left(-D(-1) + \sqrt{D(-1)^2 + 4\gamma^4\lambda^4}\right)$

(2.20) $$\psi(-1) = \ell n\left[\frac{2\gamma^2\lambda^2}{-D(-1) + \sqrt{D(-1)^2 + 4\gamma^4\lambda^4}}\right] + \frac{U_A}{U_T}$$

(2.21) $$\psi(1) = \ell n\left[\frac{-D(1) + \sqrt{D(1)^2 + 4\gamma^4\lambda^4}}{2\gamma^2\lambda^2}\right] + \frac{U_C}{U_T} \ .$$

For λ small the problem (2.11), (2.18), (2.19), (2.20), (2.21) constitutes a singularly perturbed two-point boundary value problem.

λ small means that $\bar{C}$ is large (assuming that ℓ is constant). In practical cases $\bar{c} > 10^{17}$ such that $\lambda^2 < 0.4 \times 10^{-6}$ holds. For the following analysis we assume for the sake of simplicity that D(x) is independent of λ (it would suffice to assume that D is analytic in λ), that means that the doping $|C(z)|'$ increases 'as a whole' as $\lambda \to 0+$. Actually, the asymptotic analysis presented in the next sections requires that

(2.22) $$\frac{\min_{z \in [-\ell,\ell]} |D(z)|}{\max_{z \in [-\ell,\ell]} |D(z)|} \gg \lambda^2$$

and since $\gamma^2\lambda^2 = \dfrac{n_i}{\max\limits_{z \in [-\ell,\ell]} |C(z)|}$ that

(2.23) $n_i \ll \max\limits_{z \in [-\ell,\ell]} |C(z)|$

holds.

The two cases we deal with now are

(A)
$$D(x) < 0 \quad \text{for} \quad x \in [-1,X]; \; D(x) > 0 \quad \text{for} \quad x \in (X,1] \quad \text{and}$$
$$|D(x)| > D_A \quad \text{for} \quad x \in [-1,1]$$

which corresponds to the pn-junction and

(B)
$$D(x) > D_B > 0 \quad \text{for} \quad x \in [-1,1]$$

corresponding to an nn^+ or n^+n junction. In both cases (A) and (B) we assume that $D(X+)$, $D(X-)$ (we use the notation of $f(X\pm) = \lim_{x \to X\pm} f(x)$ in the sequel) exist and $D(X+) \neq D(X-)$ and that D is sufficiently smooth everywhere else.

The analysis of the scaled problem is complicated by the logarithmic blow-up of the boundary data of ψ as given by (2.20), (2.21) in the case (A).

The potential difference of the contacts is given by

$$(2.24) \qquad \psi(-1) - \psi(1) = \frac{U}{U_T} + \frac{U_{bi}(\lambda)}{U_T}$$

where the build-in-voltage $U_{bi}(\lambda)$ (i.e. the voltage due to doping) is given by

$$(2.25) \qquad \frac{U_{bi}(\lambda)}{U_T} = \ln\left[\frac{D(-1) + \sqrt{D(-1)^2 + 4\gamma^4\lambda^4}}{D(1) + \sqrt{D(1)^2 + 4\gamma^4\lambda^4}}\right] = \begin{cases} \ln\left(\frac{\gamma^4\lambda^4}{D(1)|D(-1)|}\right) + 0(\gamma^4\lambda^4), \; D(-1) < 0 \;\; (A) \\[2ex] \ln\left(\frac{D(-1)}{D(1)}\right) + 0(\gamma^4\lambda^4), \; D(-1) > 0 \;\; (B) \;\;. \end{cases}$$

$U_{bi}(\lambda)$ is bounded as $\lambda \to 0+$ in the case (B). Since (2.11) depends only on ψ', ψ'' (and not on ψ) we can therefore remove the singularity in the boundary conditions by substituting ψ by $\psi - \ln\frac{1}{\gamma^2\lambda^2}$. The equations (2.11) remain unchanged and the new boundary conditions for the case (B) are:

$$(2.26) \qquad \psi(-1) = \ln[\frac{1}{2}(D(-1) + \sqrt{D(-1)^2 + 4\gamma^4\lambda^4})] + \frac{U_A}{U_T}$$

$$(2.27) \qquad \psi(1) = \ln[\frac{1}{2}(D(1) + \sqrt{D(1)^2 + 4\gamma^4\lambda^4})] + \frac{U_C}{U_T} \;\;.$$

-10-

3. Expansions

In this Section we apply the standard approach for singularly perturbed boundary value problems to the semiconductor problem. We assume that the solution has a formal asymptotic expansion in λ, each term in the series being the sum of a uniformly smooth function and layer terms.

A problem that occurs is the blow up of the boundary values of ψ in the case (A), which implies that 'reduced' boundary conditions ($\lambda=0$) cannot be defined formally for ψ. For the derivation of the expansions we set in the case (A)

(3.1)(a) $\qquad\qquad$ (a) $\psi(-1) = \psi_-$, $\qquad$ (b) $\psi(1) = \psi_+$

and assume that ψ_-, ψ_+ are independent of λ (this will be justified later). In the case (B) 'reduced' boundary conditions for ψ can be obtained from (2.26), (2.27) and we set:

(3.1) $\qquad\qquad$ (c) $\psi_- = \ln D(-1) + \dfrac{U_A}{U_T}$, $\qquad$ (d) $\psi_+ = \ln D(1) + \dfrac{U_C}{U_T}$.

We make the following 'ansatz':

(3.2)(a) $\qquad\qquad \psi(x,\lambda) \sim \bar{\psi}(x) + \hat{\psi}(\sigma) + \tilde{\psi}_\ell(\tau) + \tilde{\psi}_r(\phi) + \cdots$

(3.2)(b) $\qquad\qquad n(x,\lambda) \sim \bar{n}(x) + \hat{n}(\sigma) + \tilde{n}_\ell(\tau) + \tilde{n}_r(\phi) + \cdots$

(3.2)(c) $\qquad\qquad p(x,\lambda) \sim \bar{p}(x) + \hat{p}(\sigma) + \tilde{p}_\ell(\tau) + \tilde{p}_r(\phi) + \cdots$

(3.2)(d) $\qquad\qquad J_n(x,\lambda) \sim \bar{J}_n(x) + \hat{J}_n(\sigma) + \tilde{J}_{n_\ell}(\tau) + \tilde{J}_{n_r}(\phi) + \cdots$

(3.2)(e) $\qquad\qquad J_p(x,\lambda) \sim \bar{J}_p(x) + \hat{J}_p(\sigma) + \tilde{J}_{p_\ell}(\tau) + \tilde{J}_{p_r}(\phi) + \cdots$

where the dots stand for a power series in λ (starting with the $O(\lambda)$-term) whose coefficients are of the same form as the given $O(1)$ terms. The fast variables are

(3.3) $\qquad\qquad$ (a) $\sigma = \dfrac{x-X}{\lambda}$

(3.3) $\qquad\qquad$ (b) $\tau = \dfrac{x+1}{\lambda}$ $\quad\Big\}$ $\quad x \in [-1,1]$

(3.3) $\qquad\qquad$ (c) $\phi = \dfrac{x-1}{\lambda}$

The functions marked with '-' denote the reduced solution (of order zero), '^' denotes the internal layer terms (at $x = X$) (of order zero), '~' denotes the layer terms (of order zero) decaying from the left boundary $x = -1$ (with index ℓ) and the layer terms (of order zero) decaying from the right boundary $x = +1$ (with index r) resp. The boundary condition

(3.4)   (a) $\hat{\psi}(\pm\infty) = \hat{n}(\pm\infty) = \hat{p}(\pm\infty) = \hat{J}_n(\pm\infty) = \hat{J}_p(\pm\infty) = 0$

(3.4)   (b) $\tilde{\psi}_\ell(\infty) = \tilde{n}_\ell(\infty) = \tilde{p}_\ell(\infty) = \tilde{J}_n(\infty) = \tilde{J}_p(\infty) = 0$

(3.4)   (c) $\tilde{\psi}_r(-\infty) = \tilde{n}_r(-\infty) = \tilde{p}_r(-\infty) = \tilde{J}_n(-\infty) = \tilde{J}_p(-\infty) = 0$

hold, since the internal layer terms are regarded as functions on $\mathbb{R}$, the left layer terms are regarded as functions on $[0,\infty)$, the right layer terms as functions on $(-\infty,0]$.

We assume that S_n, $S_p \in C^1([0,\infty)^2 \times \mathbb{R}^2 \times [0,\gamma\lambda_0])$ and that $\lambda < \lambda_0$. Inserting into (2.11), comparing $0(1)$-terms and evaluating away from $x = \pm1$, X gives the reduced problem (or order zero):

$$
(3.5) \quad
\left.
\begin{array}{ll}
(a) & 0 = \bar{n} - \bar{p} - D(x) \\[4pt]
(b) & \bar{n}' = \bar{n}\,\bar{\psi}' + \bar{J}_n \\[4pt]
(c) & \bar{p} = -\bar{p}\,\bar{\psi}' - \bar{J}_p \\[4pt]
(d) & \bar{J}_n' = S_n(\bar{n},\bar{p},\bar{J}_n,\bar{J}_p,0) \\[4pt]
(e) & \bar{J}_p' = -S_p(\bar{n},\bar{p},\bar{J}_n,\bar{J}_p,0)
\end{array}
\right\} \quad -1 < x < 1 \ .
$$

We have to expect that $\bar{\psi}, \bar{n}, \bar{p}$ are discontinuous at $x = X$, therefore (3.5) has to hold for the right hand and left hand limits at $x = X$. Evaluation close to $X+$ gives the right (zeroth order) (internal) layer problem

$$
(3.6) \quad
\left.
\begin{array}{ll}
(a) & \ddot{\hat{\psi}} = \hat{n} - \hat{p} \\[6pt]
(b) & \dot{\hat{n}} = (\hat{n} + \bar{n}(X+))\dot{\hat{\psi}} \\[6pt]
(c) & \dot{\hat{p}} = -(\hat{p} + \bar{p}(X+))\dot{\hat{\psi}} \\[6pt]
(d) & \dot{\hat{J}}_n = 0 \\[6pt]
(e) & \hat{J}_p = 0
\end{array}
\right\} \quad 0 < \sigma < \infty
$$

-12-

('•' denotes differentaition with respect to the corresponding fast variable in the sequel) and evaluation close to $X-$ gives the left (zeroth order) (internal) layer problem

(3.7)

(a) $\ddot{\hat{\psi}} = \hat{n} - \hat{p}$

(b) $\ddot{\hat{n}} = (\hat{n} + \bar{n}(X-))\dot{\hat{\psi}}$

(c) $\ddot{\hat{p}} = -(\hat{p} + \bar{p}(X-))\dot{\hat{\psi}}$ $-\infty < \sigma < 0$.

(d) $\dot{\hat{J}}_n = 0$

(e) $\dot{\hat{J}}_p = 0$

Similarly we obtain the left (boundary) layer problem

(3.8)

(a) $\ddot{\tilde{\psi}}_\ell = \tilde{n}_\ell - \tilde{p}_\ell$

(b) $\dot{\tilde{n}}_\ell = (\tilde{n}_\ell + \bar{n}(-1))\dot{\tilde{\psi}}_\ell$

(c) $\dot{\tilde{p}}_\ell = -(\tilde{p}_\ell + \bar{p}(-1))\dot{\tilde{\psi}}_\ell$ $0 < \tau < \infty$.

(d) $\dot{\tilde{J}}_{n_\ell} = 0$

(e) $\dot{\tilde{J}}_{p_\ell} = 0$

The right (boundary) layer problem is obtained from (3.8) by substituting $\bar{n}(-1)$, $\bar{p}(-1)$ by $\bar{n}(1)$, $\bar{p}(1)$.

Inserting into (2.19), (3.1)(a) and comparing $0(1)$-coefficients of λ gives the matching conditions at $x = -1$

(3.9)

(a) $\bar{n}(-1) + \tilde{n}_\ell(0) = \begin{cases} D(-1) & , \; D(-1) > 0 \\ 0 & , \; D(-1) < 0 \end{cases}$

(b) $\bar{p}(-1) + \tilde{p}_\ell(0) = \begin{cases} 0 & , \; D(-1) > 0 \\ -D(-1) & , \; D(-1) < 0 \end{cases}$

(c) $\bar{\psi}(-1) + \tilde{\psi}_\ell(0) = \psi_-$

(2.18), (3.1)(b) gives the matching conditions at $x = 1$

-13-

$$\text{(a)} \quad \bar{n}(1) + \tilde{n}_r(0) = D(1)$$

(3.10)
$$\text{(b)} \quad \bar{p}(1) + \tilde{p}_r(0) = 0$$

$$\text{(c)} \quad \bar{\psi}(1) + \tilde{\psi}_r(0) = \psi_+ \quad .$$

The continuity conditions (2.16) give the interface conditions

$$\text{(a)} \quad \bar{n}(X-) + \hat{n}(0-) = \bar{n}(X+) + \hat{n}(0+)$$

$$\text{(b)} \quad \bar{p}(X-) + \hat{p}(0-) = \bar{p}(X+) + \hat{p}(0+)$$

$$\text{(c)} \quad \bar{\psi}(X-) + \hat{\psi}(0-) = \bar{\psi}(X+) + \hat{\psi}(0+)$$

(3.11)
$$\text{(d)} \quad \overset{\mathit{l}}{\psi}(0-) = \overset{\mathit{l}}{\psi}(0+)$$

$$\text{(e)} \quad \bar{J}_n(X-) + \hat{J}_n(0-) = \bar{J}_n(X+) + \hat{J}_n(0+)$$

$$\text{(f)} \quad \bar{J}_p(X-) + \hat{J}_p(0-) = \bar{J}_p(X+) + \hat{J}_p(0+) \quad .$$

From (3.7)(d),(e), (3.8)(d),(e) and from the analogous equations for the right J_n, J_p-layer terms we immediately conclude that

(3.12)
$$\hat{J}_n \equiv \hat{J}_p \equiv 0, \ \tilde{J}_{n_{\mathit{l}}} \equiv \tilde{J}_{p_{\mathit{l}}} \equiv 0, \ \tilde{J}_{n_r} \equiv \tilde{J}_{p_r} \equiv 0$$

since (3.4) has to hold. No zeroth order layers occur in J_n, J_p. The current densities are the slow components.

The problem (3.8), (3.9) has been dealt with in Markowich, Ringhofer, et al (1982) and it has been shown that

(3.13)
$$\tilde{\psi}_{\mathit{l}} \equiv \tilde{n}_{\mathit{l}} \equiv \tilde{p}_{\mathit{l}} \equiv 0$$

holds. The same analysis goes throught for the right boundary layer terms and

(3.14)
$$\tilde{\psi}_r \equiv \tilde{n}_r \equiv \tilde{p}_r \equiv 0$$

follows. No zeroth order boundary layers occur, since the reduced boundary conditions for n, p ((2.18), (2.19) with $\lambda = 0$) can be fulfilled by the reduced solution due to (3.5)(a).

By including more terms in the expansion (3.2) it turns out that higher order boundary layer terms occur. Similarly, higher order internal layers occur in the slow component $J_n(J_p)$ if $\frac{\partial S_n}{\partial n}$ or $\frac{\partial S_n}{\partial p}$ $(\frac{\partial S_p}{\partial n}$ or $\frac{\partial S_p}{\partial n})$ are not constant zero.

-14-

Integrating (3.6)(b),(c), (3.7)(b),(c) gives

$$(3.15) \qquad \hat{n}(\sigma) = \begin{cases} \bar{n}(X+)(e^{\hat{\psi}(\sigma)}-1) & , \quad \sigma > 0 \\[2mm] \bar{n}(X-)(e^{\hat{\psi}(\sigma)}-1) & , \quad \sigma < 0 \end{cases}$$

$$(3.16) \qquad \hat{p}(\sigma) = \begin{cases} \bar{p}(X+)(e^{-\hat{\psi}(\sigma)}-1) & , \quad \sigma > 0 \\[2mm] \bar{p}(X-)(e^{-\hat{\psi}(\sigma)}-1) & , \quad \sigma < 0 \end{cases}$$

Inserting (3.15), (3.16) into (3.11)(a),(b) gives

$$(3.17) \qquad \bar{n}(X-)e^{\hat{\psi}(0-)} = \bar{n}(X+)e^{\hat{\psi}(0+)}$$

$$(3.18) \qquad \bar{p}(X-)e^{-\hat{\psi}(0-)} = \bar{p}(X+)e^{-\hat{\psi}(0+)} \quad .$$

From (3.11)(c), (3.5)(a) we get the interface conditions for the reduced problem

$$(3.19) \qquad \text{(a)} \quad \bar{n}(X+)e^{\bar{\psi}(X-)-\bar{\psi}(X+)} = \bar{n}(X-)$$

$$\text{(b)} \quad (\bar{n}(X-) - D(X-))e^{\bar{\psi}(X-)-\bar{\psi}(X+)} = n(X+) - D(X+)$$

and from (3.11)(e),(f) and (3.12)

$$(3.19) \qquad \text{(c)} \quad \bar{J}_n(X+) = \bar{J}_n(X-)$$

$$\text{(d)} \quad \bar{J}_p(X+) = \bar{J}_p(X-) \quad .$$

The boundary conditions follow from (3.9), (3.10) and (3.13), (3.14)

$$(3.20) \qquad \text{(a)} \quad \bar{n}(-1) = \begin{cases} 0 & , \quad D(-1) < 0 \\[2mm] D(-1) & , \quad D(-1) > 0 \end{cases}$$

$$\text{(b)} \quad \bar{\psi}(-1) = \psi_-$$

$$(3.21) \qquad \text{(a)} \quad \bar{n}(1) = D(1)$$

$$\text{(b)} \quad \bar{\psi}(1) = \psi_+ \quad .$$

Eliminating $\bar{p}$ from (3.5)(b),(c) using (3.5)(a) gives the reduced equations

$$\text{(a)} \quad \bar{\psi}' = \frac{D' - (\bar{J}_n + \bar{J}_p)}{2\bar{n} - D}$$

$$\text{(b)} \quad \bar{n}' = \frac{(\bar{n}-D)\bar{J}_n - \bar{n}\bar{J}_p + \bar{n}D'}{2\bar{n} - D} \quad \left. \begin{array}{l} -1 < x < X \\ \text{and} \\ X < x < 1 \end{array} \right.$$

(3.22)

$$\text{(c)} \quad \bar{J}_n' = S_n(\bar{n},\bar{p},\bar{J}_n,\bar{J}_p,0)$$

$$\text{(d)} \quad \bar{J}_p' = -S_p(\bar{n},\bar{p},\bar{J}_n,\bar{J}_p,0)$$

assuming that $D \in C^1([-1,X]) \cap C^1([X,1])$. $\bar{p}$ is given by (3.5)(a):

(3.22) $\text{(e)} \quad \bar{p} = \bar{n} - D(x)$.

We obtain the internal layer problem by inserting (3.15), (3.16) into (3.6)(a), (3.7)(a):

$$\text{(a)} \quad \ddot{\hat{\psi}} = \bar{n}(X-)e^{\hat{\psi}} - \bar{p}(X-)e^{-\hat{\psi}} - D(X-), \quad -\infty < \sigma < 0$$

(3.23)

$$\text{(b)} \quad \ddot{\hat{\psi}} = \bar{n}(X+)e^{\hat{\psi}} - \bar{p}(X+)e^{-\hat{\psi}} - D(X+), \quad 0 < \sigma < \infty$$

subject to the boundary conditions

$$\text{(c)} \quad \hat{\psi}(-\infty) = 0$$

(3.23)

$$\text{(d)} \quad \hat{\psi}(\infty) = 0$$

and the interface conditions

$$\text{(e)} \quad \hat{\psi}(0+) - \hat{\psi}(0-) = \bar{\psi}(X-) - \bar{\psi}(X+)$$

(3.23)

$$\text{(f)} \quad \dot{\hat{\psi}}(0+) = \dot{\hat{\psi}}(0-) .$$

Because of (2.17) we require that the solutions $\bar{n}$, $\bar{p}$ of the reduced problem are nonnegative, that means:

(3.24) $\bar{n}(x) \geqslant \max(0, D(x)), \ x \in [-1,1]$

has to hold. Under this assumption we prove a simple consequence of the interface conditions (3.19).

<u>Lemma 3.1.</u> Assume that (3.24) holds (at least at $X-$, $X+$). Then

(3.25)
$$D(X-) < 0, \ D(X+) > 0 \Longrightarrow$$
$$(\bar{\psi}(X-) < \bar{\psi}(X+) \ \text{ and } \ \bar{n}(X+) > D(X+), \ \bar{n}(X-) > 0)$$

holds.

-16-

Proof: Assume first that $\bar{\psi}(X-) = \bar{\psi}(X+)$. Then (3.19)(a) implies that $\bar{n}(X+) = \bar{n}(X-)$ and (3.19)(b) implies $D(X-) = D(X+)$. This is a contradiction to the assumption that D has a jump discontinuity at $x = X$. Therefore $\bar{\psi}(X-) \neq \bar{\psi}(X+)$. We compute $\bar{n}(X+)$ from (3.19)

$$(3.26) \qquad \bar{n}(X+) = \frac{D(X+) - D(X-)e^{\bar{\psi}(X-) - \bar{\psi}(X+)}}{1 - e^{2(\bar{\psi}(X-) - \bar{\psi}(X+))}} \ .$$

(3.25) follows immediately from (3.26).

$\square$

We now give existence theorems for the reduced problem and start with the simple case (B).

Theorem 3.1. Assume that $D(x) > D_B > 0$ on $[-1,1]$, $D \in C^1([-1,X])$, $D \in C^1([X,1])$ and that

$$(3.27) \qquad S_n(D(x),0,J,0,0) \equiv S_p(D(x),0,J,0,0) \equiv 0 \quad \text{for all } x \in [-1,1]$$

$$\text{and all } J \in \mathbb{R} \ .$$

Then the reduced problem (3.22), (3.19), (3.20), (3.21) has the solution

$$
\begin{aligned}
&\text{(a)} \quad \bar{n}(x) \equiv D(x) \\
&\text{(b)} \quad \bar{p}(x) \equiv 0 \\
&\text{(c)} \quad \bar{J}_p(x) \equiv 0 \\
&\text{(d)} \quad \bar{J}_n(x) \equiv \frac{U}{U_T} \Big/ \int_{-1}^{1} \frac{ds}{D(s)}
\end{aligned}
\quad
\left.\begin{aligned} \\ \\ \\ \\ \end{aligned}\right\}
\begin{aligned} -1 &< x < X \\ X &< x < 1 \end{aligned}
$$

$$(3.28)$$

$$
\text{(e)} \quad \bar{\psi}(x) =
\begin{cases}
\dfrac{U_A}{U_T} + \ln D(x) - \dfrac{U}{U_T} \dfrac{\int_{-1}^{x} \frac{ds}{D(s)}}{\int_{-1}^{1} \frac{ds}{D(s)}} \ , & -1 < x < X \\[3ex]
\dfrac{U_C}{U_T} + \ln D(x) - \dfrac{U}{U_T} \dfrac{\int_{+1}^{x} \frac{ds}{D(s)}}{\int_{-1}^{1} \frac{ds}{D(s)}} \ , & X < x < 1 \ .
\end{cases}
$$

Proof: Assume (3.28)(a),(b),(c). Then (3.27), (3.19)(c) imply that $\bar{J}_n \equiv \text{const}$ on $[-1,1]$. From (3.22)(a) we conclude that

-17-

$$\bar{\psi}(x) = \begin{cases} \psi_- + \ln \frac{D(x)}{D(-1)} - \bar{J}_n \int_{-1}^{x} \frac{ds}{D(s)} \ , & -1 < x < X \\ \psi_+ + \ln \frac{D(x)}{D(1)} - \bar{J}_n \int_{1}^{x} \frac{ds}{D(s)} \ , & X < x < 1 \end{cases}$$

holds. Now (3.22) and all boundary conditions are fulfilled. $\bar{J}_n$ has to be calculated from (3.19)(a). ((3.19)(b) is automatically fulfilled.) (3.28)(e) follows then by using (3.1)(c),(d).

$\square$

For a recombination rate R which depends only on n, p, (3.27) is a direct consequence of the equilibrium condition. Therefore Theorem 3.1 holds for the SRH-recombination term.

Assuming the validity of the asymptotic expansions (3.2) (which will be proven later) the theorem implies that the device is depleted of holes (away from the junction) and that the electron current J_n is asymptotically proportional to the applied voltage U. Actually (3.28)(d) is a scaled version of Ohm's law $(\int_{-1}^{1} \frac{ds}{D(s)}$ is the (scaled) resistance of the device). n^+n and nn^+ junctions are resistors.

Now we turn to the case (A). For simplicity we take the SRH-recombination term.

__Theorem 3.2.__ Assume that $D(x) < 0$ on $[-1,X]$, $D(x) > 0$ on $[X,1]$, $|D(x)| > D_A$ on $[-1,1]$ and that $D \in C^1(]-1,X])$, $D \in C^1([X,1])$. Let S_n, S_p be given by (2.14) (SRH). Moreover assume that

(3.29) $\qquad\qquad\qquad e^{\psi_- - \psi_+} < \rho, \ \rho$ sufficiently small

holds. Then the reduced problem (3.22), (3.19), (3.20), (3.21) has a locally unique solution (in $C^1(([-1,X) \cup (X,1])^4)$ which fulfills (3.24) and

$$\text{(a)} \quad \bar{\psi}(x) = \begin{cases} \psi_- - \ln \dfrac{D(x)}{D(-1)} + \bar{\phi}(x), & -1 \leqslant < X \\[2em] \psi_+ + \ln \dfrac{D(x)}{D(1)} + \bar{\phi}(x), & X < x \leqslant 1 \end{cases}$$

$$\bar{\phi}(x) = O(e^{\psi_- - \psi_+}) \quad \text{on} \quad [-1,1]$$

(3.30)

$$\text{(b)} \quad \bar{n}(x) = \begin{cases} O(e^{\psi_- - \psi_+}) > 0 & , \quad -1 \leqslant x < X \\[2em] D(x) + O(e^{\psi_- - \psi_+}) & , \quad X < x \leqslant 1 \end{cases}$$

$$\text{(c)} \quad \bar{J}_n(x) = O(e^{\psi_- - \psi_+}) \quad , \quad -1 \leqslant x \leqslant 1$$

$$\text{(d)} \quad \bar{J}_p(x) = O(e^{\psi_- - \psi_+}) \quad , \quad -1 \leqslant x \leqslant 1 \; .$$

Of course $\bar{p}(x)$ is then given by

(3.30)

$$\text{(e)} \quad \bar{p}(x) = \begin{cases} -D(x) + O(e^{\psi_- - \psi_+}) & , \quad -1 \leqslant x < X \\[2em] O(e^{\psi_- - \psi_+}) > 0 & , \quad X < x \leqslant 1 \; . \end{cases}$$

<u>Proof</u>: We introduce $\bar{\phi}$ as a new dependent variable (instead of $\bar{\psi}$) and obtain from (3.22)(a)

$$\text{(a)} \quad \bar{\phi}' = \frac{2DD' - (\bar{J}_n + \bar{J}_p)D - 2\bar{n}D'}{D(2\bar{n}-D)} \quad , \quad X < x \leqslant 1$$

(3.31)

$$\text{(b)} \quad \bar{\phi}' = \frac{2\bar{n}D' - D(\bar{J}_n + \bar{J}_p)}{D(2\bar{n}-D)} \quad , \quad -1 \leqslant x < X \; .$$

Boundary conditions for $\bar{\phi}$ are

(3.32) $$\bar{\phi}(1) = \bar{\phi}(-1) = 0$$

and the interface conditions (3.19)(a),(b) transform to

$$\text{(a)} \quad e^{\psi_- - \psi_+} \frac{D(-1)D(1)}{D(X-)D(X+)} e^{\bar{\phi}(X-) - \bar{\phi}(X+)} \bar{n}(X+) - \bar{n}(X-) = 0$$

(3.33)

$$\text{(b)} \quad e^{\psi_- - \psi_+} \frac{D(-1)D(1)}{D(X-)D(X+)} e^{\bar{\phi}(X-) - \bar{\phi}(X+)} (\bar{n}(X-) - D(X-)) - (\bar{n}(X+) - D(X+)) = 0 \; .$$

-19-

We set $w = e^{\psi_- - \psi_+}$ and $\bar{z}_0 = (\bar{\phi}_0, \bar{n}_0, \bar{J}_{n_0}, \bar{J}_{p_0})$, where

$$\bar{\phi}_0 \equiv 0 \quad \text{on} \quad [-1,1]$$

(3.34)
$$\bar{n}_0(x) = \begin{cases} 0 & , \quad -1 \leqslant x \leqslant X \\ D(x), & X < x \leqslant 1 \end{cases}$$

$$\bar{J}_{n_0}(x) \equiv \bar{J}_{p_0}(x) \equiv 0 \quad \text{on} \quad [-1,1]$$

and write the problem (3.31), (3.22)(b) – (d), (3.20)(a), (3.21)(a), (3.32), (3.33), (3.19)(c),(d) in operator form $F(w,\bar{z}) = 0$ where $F : [0,\infty) \times (C_X^1([-1,1])^4 \to (C_X([-1,1])^4 \times \mathbb{R}^8)$. $C_X^i([-1,1])$ is the space of functions of which are i-times continuously differentiable on $[-1,X)$ and on $(X,1]$ and $\lim_{x \to X\pm} f^{(j)}(x)$ exists for $0 \leqslant j \leqslant i$. The space is equipped with the norm $\|f\|_{X,i} = \sum_{j=0}^{i} \sup_{\substack{x \in [-1,1] \\ x \neq X}} |f^{(j)}(x)|$.

Obviously $F(0,\bar{z}_0) = 0$.

We investigate the equation $D_z F(0,\bar{z}_0)y = (f,\alpha)$ (where $D_z F(0,\bar{z}_0)$ denotes the Fréchet derivative of F with respect to z at $(0,\bar{z}_0)$)   for $y = (y_1,y_2,y_3,y_4)$ and obtain

(3.35) (a) $y' = \begin{bmatrix} 0 & -\dfrac{2D'}{D^2} & -\dfrac{1}{D} & -\dfrac{1}{D} \\[2ex] 0 & -\dfrac{D'}{D} & 0 & -1 \\[2ex] 0 & \dfrac{1}{\beta} & 0 & 0 \\[2ex] 0 & -\dfrac{1}{\beta} & 0 & 0 \end{bmatrix} y + f, \quad X < x < 1$

(3.35) (b) $y' = \begin{bmatrix} 0 & -\dfrac{2D'}{D^2} & -\dfrac{1}{D} & -\dfrac{1}{D} \\[2ex] 0 & -\dfrac{D'}{D} & 1 & 0 \\[2ex] 0 & \dfrac{1}{\beta} & 0 & 0 \\[2ex] 0 & -\dfrac{1}{\beta} & 0 & 0 \end{bmatrix} y + f, \quad -1 < x < X$

-20-

where $f \in (C_x([-1,1]))^4$. The boundary and interface conditions are

$$(c) \quad y_1(-1) = \alpha_1, \quad y_2(-1) = \alpha_2$$

$$(d) \quad y_1(1) = \alpha_3, \quad y_2(1) = \alpha_4$$

$$(e) \quad y_3(X+) - y_3(X-) = \alpha_5$$

(3.35)

$$(f) \quad y_4(X+) - y_4(X-) = \alpha_6$$

$$(g) \quad y_2(X-) = \alpha_7$$

$$(h) \quad y_2(X+) = \alpha_8$$

with $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, 8$.

Because of the Fredholm alternative $D_z F(0, \bar{z}_0)$ is one-to-one and onto iff it is one-to-one. Therefore we only have to show that the homogeous problem (3.35) ($\alpha = 0$, $f \equiv 0$) has the unique solution $y \equiv 0$.

From (3.35)(a) we get

$$\left.\begin{aligned} y_2' &= -\frac{D'}{D} y_2 - y_4 \\[2ex] y_3' &= \frac{1}{\beta} y_2 \\[2ex] y_4' &= -\frac{1}{\beta} y_2 \; . \end{aligned}\right\} \quad X < x < 1$$

(3.36)

Therefore

$$y_4'' = -\frac{1}{\beta} y_2' = -\frac{1}{\beta} \left(-\frac{D'}{D} y_2 - y_4 \right) = -\frac{1}{\beta} \left(\beta \frac{D'}{D} y_4' - y_4 \right)$$

and

(3.37)(a)
$$y_4'' = (\ln D)' y_4' - \frac{1}{\beta} y_4 = 0, \quad X < x < 1$$

holds. The boundary conditions are

(3.37)(b)
$$y_4'(1) = y_4'(X+) = 0 \; .$$

Since $\beta > 0$ the maximum principle implies that $y_4 \equiv 0$ on $[X,1]$ and therefore $y_3 \equiv y_2 \equiv y_1 \equiv 0$ on $[X,1]$.

-21-

(3.35)(b) gives

$$y_2' = -\frac{D'}{D} y_2 + y_3$$

(3.38) $\qquad\qquad y_3' = \frac{1}{\beta} y_2 \qquad \left.\right\} \; -1 < x < X$

$$y_4' = -\frac{1}{\beta} y_2 \;\; .$$

Again we get a second order problem

(3.39)(a) $\qquad\qquad y_3'' + (\ell n \; |D|)' y_3' - \frac{1}{\beta} y_3 = 0, \; -1 < x < X$

(3.39)(b) $\qquad\qquad y_3'(-1) = y_3'(X-) = 0$

and the maximum principle yields $y_3 \equiv 0$ on $[-1,X]$. $y_1 \equiv y_2 \equiv y_3 \equiv 0$ on $[-1,X]$

follows immediately.

Therefore $D_z F(0,\bar{z}_0)$ is an isomorphism and since $D_z F(w,\bar{z}_0)$ is uniformly Lipschitz

continuous the implicit function theorem assures that there is a locally unique solution

$\bar{z} = \bar{z}(w)$ of $F(w,\bar{z}) = 0$ for $w \in [0,w_0$ sufficiently small. Since

$$F(w,\bar{z}_0) = O(w)$$

we get $\|\bar{z}(w) - \bar{z}_0\|_{X,1} = O(w)$. To show that this solution $\bar{z}(w)$ fulfills (3.24) we

compute the first order term $\bar{z}_1$ of the expansion

$$\bar{z}(w) \sim \sum_{i=0}^{\infty} w^i \bar{z}_i$$

as a solution of the equation

$$D_z F(0,\bar{z}_0)z_1 = -D_w F(0,\bar{z}_0) \;\; .$$

$\bar{z}_1$ solves (3.35)(a),(b) with $f \equiv 0$, fulfills the interface conditions (setting

$\bar{z}_1 = (\bar{\psi}_1, \bar{n}_1, \bar{J}_{n_1}, \bar{J}_{p_1}))$

$$\overline{n}_1(X-) = \frac{D(-1)D(1)}{D(X-)} > 0$$

$$\overline{n}_1(X+) = -\frac{D(-1)D(1)}{D(X+)} > 0$$

$$\overline{J}_{n_1}(X+) - \overline{J}_{n_1}(X-) = 0$$

$$\overline{J}_{p_1}(X+) - \overline{J}_{p_1}(X-) = 0$$

and the boundary conditions

$$\overline{\psi}_1(-1) = \overline{\psi}_1(1) = 0$$

$$\overline{n}_1(-1) = \overline{n}_1(1) = 0 \quad .$$

$\overline{n}_1$, $\overline{J}_{n_1}$, $\overline{J}_{p_1}$ fulfill (3.36) and (3.38) and therefore $\overline{J}_{p_1}$ fulfills (3.37)(a) on $[X,1]$ subject to the boundary conditions

$$\overline{J}'_{p_1}(1) = 0, \quad \overline{J}'_{p_1}(X+) = \frac{1}{\beta}\frac{D(-1)D(1)}{D(X+)} < 0 \quad .$$

The maximum principle implies that $\overline{J}'_{p_1} = -\frac{1}{\beta}\overline{n}_1$ is negative on $[X,1)$, such that $\overline{n}_1 > 0$ on $[X,1)$ holds.

Similarly we obtain $\overline{n}_1 > 0$ on $(-1,X]$. Since the zeros $x = +1, -1$ of $\overline{n}_1$ are simple zeros, we obtain (3.24).

$\square$

The biggest restriction of the Theorem 3.2 is the required smallness of $e^{\psi_- - \psi_+}$. We obtain from (3.1)(a) and (2.24)

(3.40)
$$e^{\psi_- - \psi_+} = e^{\frac{U}{U_T} + \frac{U_{bi}(\lambda)}{U_T}} \quad .$$

Therefore (3.29) holds if there is a constant $K > 0$ sufficiently large (but independent of λ) such that

(3.41)
$$\frac{U}{U_T} < \frac{|U_{bi}(\lambda)|}{U_T} - K$$

holds. The applied voltage U has to be sufficiently smaller than the absolute value of the built-in voltage (low-injection condition).

Numerical calculations (given in Section 5) demonstrate that (3.41) is <u>not</u> necessary for the existence of a reduced solution.

From (3.30)(c)(d) we get the reduced voltage-current characteristic

$$(3.42) \qquad |\bar{J}_n| + \bar{J}_p| = 0 \left(\frac{\gamma^4 \lambda^4}{D(1)|D(-1)|} e^{\frac{U}{U_T}} (1 + O(\gamma^4 \lambda^4)) \right) \ .$$

In the case (A) (pn-junction) the total current density depends exponentially on the applied voltage (see also Sze (1969)). (3.42) should be compared to the corresponding result (3.28)(c),(d) for the case (B).

Theorem 3.2 can easily be generalized to more general functions S_n, S_p which do not depend on J_n, J_p. (3.30) holds without change for $S_n \equiv S_p \equiv 0$.

Now we turn to the internal layer problem (3.23). We prove

<u>Theorem 3.3</u>: <u>Case A</u>: Set $D(X+) > 0$, $D(X-) < 0$. Then, if the reduced problem (3.22), (3.19), (3.20), (3.21) has a solution fulfilling (3.24), the internal layer problem (3.23) has a unique piecewise monotone solution $\hat{\psi}$ which fulfills:

(a)
$$0 < \hat{\psi}(\sigma) < C_\delta \exp\left((1-\delta)\sqrt{\overline{n}(X-) + \overline{p}(X-)} \ \sigma + D_\delta \sqrt{\overline{\psi}(X+) - \overline{\psi}(X-)} \right)$$

$$\text{for} \quad \sigma < -E_\delta \sqrt{\overline{\psi}(X+) - \overline{\psi}(X-)}$$

(b)
$$0 < -\hat{\psi}(\sigma) < C_\delta \exp\left((-1+\delta)\sqrt{\overline{n}(X+) + \overline{p}(X+)} \ \sigma + D_\delta \sqrt{\overline{\psi}(X+) - \overline{\psi}(X-)} \right)$$

$$\text{for} \quad \sigma > E_\delta \sqrt{\overline{\psi}(X+) - \overline{\psi}(X-)}$$

for every $\delta > 0$ where $C_\delta > 0$, $D_\delta > 0$, $E_\delta > 0$ depend on δ but not on $\overline{\psi}(X\pm)$ if $\overline{\psi}(X+) - \overline{\psi}(X-)$ is sufficiently large.

<u>Case B</u>: Let D fulfill the assumptions of Theorem 3.1 and let the reduced solution be given by (3.28). Then the internal layer problem (3.23) has a unique piecewise monotone solution $\hat{\psi}$ which fulfills

-24-

$$\text{(a)} \quad |\hat{\psi}(\sigma)| \leq C_\delta \exp((-1-\delta)\sqrt{D(X-)}\ \sigma), \quad \sigma < 0$$

(3.44)

$$\text{(b)} \quad |\hat{\psi}(\sigma)| \leq C_\delta \exp((-1+\delta)\sqrt{D(X+)}\ \sigma), \quad \sigma > 0$$

for $\delta > 0$, where C_δ only depends on δ.

Piecewise monotone means monotone on $(-\infty,0)$ and on $(0,\infty)$.

<u>Proof</u>: For any piecewise monotone solution $\hat{\psi}$ of (3.23)

(3.45) $\qquad$ (a) $\operatorname{sgn} \hat{\psi}(0+) = -\operatorname{sgn} \hat{\psi}(0-)$, $\quad$ (b) $\hat{\psi}(0+) \neq 0$, $\hat{\psi}(0-) \neq 0$

has to hold. This follows from the monotonicity and from (3.23)(c),(d),(f) since $\ddot{\hat{\psi}}(0+) = \ddot{\hat{\psi}}(0-) = 0$ would imply $\hat{\psi} \equiv 0$ (because $\bar{n}(X\pm) - \bar{p}(X\pm) = D(X\pm)$ holds) which contradicts (3.23)(e) because of (3.25) and (3.28). $\hat{\psi}(0+) = 0$ (or $\hat{\psi}(0-) = 0$) contradicts (3.23)(e), too.

Only two possible cases remain:

(I) $\qquad\qquad\qquad\qquad \hat{\psi}(0+) > 0, \ \hat{\psi}(0-) < 0$

(II) $\qquad\qquad\qquad\qquad \hat{\psi}(0-) < 0, \ \hat{\psi}(0+) > 0$.

In the case (I) $\hat{\psi}$ has to be monotonically increasing on $(-\infty,0)$ and on $(0,\infty)$, in the case (II) $\hat{\psi}$ is decreasing on both intervals. In the case (I) we derive from Fife (1973, Lemma 2.1) that every piecewise monotone solution of (3.23)(a),(b),(c),(d) fulfills

(3.46)(a) $\qquad\qquad\qquad \sigma = \int_{\hat{\psi}(\sigma)}^{\hat{\psi}(0+)} \dfrac{d\tau}{\sqrt{2G(\tau')}} \ , \quad \sigma > 0$

where

$$G(\tau) = \int_0^\tau (\bar{n}(X+)e^s - \bar{p}(X+)e^{-s} - D(X+))ds$$

(3.46)(b)

$$= \bar{n}(X+)(e^\tau - 1) + \bar{p}(X+)(e^{-\tau} - 1) - D(X+)\tau$$

holds ((2.23)(a) fulfills all assumptions of Lemma 2.1 in Fife (1973) because of (3.24))

and

(3.47)(a) $\qquad\qquad\qquad \sigma = \int_{\hat{\psi}(0-)}^{\hat{\psi}(\sigma)} \dfrac{d\tau}{\sqrt{2F(\tau)}} \ , \quad \sigma < 0$

with

-25-

(3.47)(b)
$$F(\tau) = \int_0^{\tau} (\bar{p}(X-)e^s - \bar{n}(X-)e^{-s} + D(X-))ds$$
$$= \bar{p}(X-)(e^{\tau}-1) + \bar{n}(X-) + D(X-)\tau$$

(also (2.3)(b) fulfills the necessary assumptions after setting $-\phi = \psi$, $\lambda = -\sigma$).

Differentiation of (3.46)(a), (3.47)(a) gives

(3.48)
$$\dot{\psi}(0+) = -\sqrt{2G(\hat{\psi}(0+))}, \quad \dot{\psi}(0-) = -\sqrt{2F(-\hat{\psi}(0-))}$$

(3.23)(e), (f) gives the equation for $\hat{\psi}(0+)$

(3.49)
$$G(\hat{\psi}(0+)) = F(\bar{\psi}(X-) - \bar{\psi}(X+) - \hat{\psi}(0+))$$

which can be solved (uniquely) by using (3.46)(b), (3.47)(b) giving

(3.50)(a)
$$\hat{\psi}(0+) = \frac{D(X-)(\bar{\psi}(X+)-\bar{\psi}(X-))+(\bar{n}(X-)-\bar{n}(X+))+(\bar{p}(X-)-\bar{p}(X+))}{D(X-)-D(X+)}$$

(3.23)(e) implies

(3.50(b)
$$\hat{\psi}(0-) = \frac{D(X+)(\bar{\psi}(X+)-\bar{\psi}(X-))+(\bar{n}(X-)-\bar{n}(X+))+(\bar{p}(X-)-\bar{p}(X+))}{D(X+)-D(X-)} .$$

In the case (II) we proceed analogously and obtain the same formulas for $\hat{\psi}(0+)$, $\hat{\psi}(0-)$.

Therefore, a unique piecewise monotone solution of (3.23) exists iff $\hat{\psi}(0+)$, $\hat{\psi}(0-)$ as given by (3.50), have appropriate sign (and are not zero).

In the case (B) (3.50), (3.28) give

(a) $\hat{\psi}(0+) = \dfrac{D(X-)\ln\dfrac{D(X+)}{D(X-)} + D(X-) - D(X+)}{D(X+)-D(X-)}$

(3.51)

(b) $\hat{\psi}(0-) = \dfrac{D(X+)\ln\dfrac{D(X+)}{D(X-)} + D(X-)-D(X+)}{D(X+)-D(X-)} .$

Setting $y = \dfrac{D(X+)}{D(X-)} > 0$ we define

(a) $\hat{\psi}(0+) = f_1(y) = \dfrac{\ln y-(y-1)}{y-1}$

(3.52)

(b) $\hat{\psi}(0-) = f_2(y) = \dfrac{y\ln y-(y-1)}{y-1} .$

Obviously $\ln y - (y-1) < 0$ holds for $y > 0$, $y \neq 1$. Also $y\ln y - (y-1) > 0$ for $y > 0$, $y \neq 1$ holds and we find that sgn $f_1(y) = -$sgn $f_2(y)$, $f_1(y) \neq 0$, $f_2(y) \neq 0$ for $y > 0$,

$y \neq 1$. We derive from (3.52)

(3.53)
(a) $\hat{\psi}(0+) > 0$, $\hat{\psi}(0-) < 0$ iff $D(X+) < D(X-)$

(b) $\hat{\psi}(0+) < 0$, $\hat{\psi}(0-) > 0$ iff $D(X+) > D(X-)$.

In the case (A) we express $\bar{n}(X\pm)$, $\bar{p}(X\pm)$ in terms of $\bar{\psi}(X\pm)$ using the interface condition

(3.19)(a), (b) (as in (3.26)) getting

$$(\bar{n}(X-)-\bar{n}(X+)) + (\bar{p}(X-)-\bar{p}(X+)) = \frac{e^{\bar{\psi}(X-)-\bar{\psi}(X+)}-1}{e^{\bar{\psi}(X-)-\bar{\psi}(X+)}+1} (D(X-) + D(X+)). \quad \text{We set} \quad z = \bar{\psi}(X-) - \bar{\psi}(X+)$$

and obtain

(3.54)
(a) $\hat{\psi}(0+) = \dfrac{h_1(z)}{D(X+)-D(X-)}$,
(b) $\hat{\psi}(0-) = \dfrac{h_2(z)}{D(X+)-D(X-)}$

where

(3.55)(a)
$$h_1(z) = D(X+)g_2(z) + D(X-)g_1(z)$$

(3.55)(b)
$$h_2(z) = D(X-)g_2(z) + D(X+)g_1(z)$$

with

(3.56)(a)
$$g_1(z) = e^z - 1 - z(e^z+1)$$

(3.56)(b)
$$g_2(z) = e^z - 1 .$$

We restrict to $z < 0$ since $z > 0$ cannot occur because of Lemma 3.1 $g_2 < 0$ for $z < 0$

and a simple computation shows that $g_1(z) > 0$ for $z < 0$. Since $D(X+) > 0$, $D(X-) < 0$

we obtain $h_1(z) > 0$ and $h_2(z) < 0$ for $z < 0$. Therefore

(3.57)
(a) $\hat{\psi}(0+) < 0$,
(b) $\hat{\psi}(0-) > 0$

follows in the case (A).

Now the existence theorem is settled in both cases, the decay statements (3.43),

(3.44) still have to be shown.

In the case (B) the equation (3.23)(b) reads

$$\ddot{\hat{\psi}} = g(\hat{\psi}) := D(X+)(e^{\hat{\psi}} - 1) .$$

$g'(0) = D(X+)$ holds and (3.44)(b) follows from Fife (1973, Lemma 2.1). (3.44)(a) is

derived in the same way.

In the case (A) we have to keep in mind that $\bar{\psi}(X+) - \bar{\psi}(X-)$ can be large (see Theorem

-27-

3.2), which implies that $\hat{\psi}(0+)$, $\hat{\psi}(0-)$ are large. Therefore we need estimates which are uniform for large $\bar{\psi}(X+) - \bar{\psi}(X-)$.

The proof follows the lines of the proof of Lemma 2.1 in Fife (1973). We set $\hat{\phi} = -\hat{\psi}$ in (2.23)(b), call the (new) right hand side of $(\hat{\psi})$ and compute $\sqrt{f'(0)} = \sqrt{\bar{n}(X+) + \bar{p}(X+)}$. It is easy to show that

$$(3.58) \qquad f(s) > (\sqrt{f'(0)} - \delta)^2 s, \quad 0 < s < \delta$$

holds for $\delta > 0$ sufficiently small. Since f is increasing we get

$$(3.59) \qquad F(\tau) = \int_0^\tau f(s)ds > \frac{(\sqrt{f'(0)} - \delta)^2 \delta^2}{2} + f(\delta)(\tau - \delta)$$

for $\gamma > \delta$. Therefore

$$(3.60) \qquad (\sqrt{f'(0)} - \delta) \int_\delta^{-\hat{\psi}(0+)} \frac{d\tau}{\sqrt{2F(\tau)}} < D_\delta \sqrt{|\hat{\psi}(0+)|}$$

holds for $|\hat{\psi}(0+)|$ sufficiently large, where $D_\delta > 0$ is independent of $\hat{\psi}(0+)$. From Fife (1973, Lemma 2.1) we obtain

$$(3.61) \qquad |\hat{\psi}(\sigma)| < \delta \exp(-(1-\delta)\sqrt{f'(0)}\sigma + D_\delta \sqrt{|\hat{\psi}(0+)|})$$

for $\sigma > \frac{D_\delta}{\sqrt{f'(0)}} \sqrt{|\hat{\psi}(0+)|}$. (3.43)(b) follows from (3.50(a). The proof of (3.43)(a) is analogous.

$\square$

Similar estimates holds for the derivatives of $\hat{\psi}$. $\hat{n}$, $\hat{p}$ have to be computed using (3.15), (3.16). In the case (B) $\hat{p} \equiv 0$ holds (since $\bar{p} \equiv 0$).

If the interface condition (3.23)(f) is changed to $\overset{\bullet\bullet}{\psi}(0+) - \overset{\bullet\bullet}{\psi}(0-) = O(\lambda)$ then the layer solution $\hat{\psi}$ changes at most by $O(\lambda \sqrt{\bar{\psi}(X+) - \bar{\psi}(X-)})$ (in the max-norm). This follows by applying the implicit function theorem to the perturbed equation (3.49). This will be needed for the existence proof in Section 4.

The width of the internal layer at $x = X$ can be computed from Theorem 3.3.

In the case (B) we obtain

-28-

$$\text{(a)} \quad d_+(\lambda) = O\left(\frac{\lambda}{\sqrt{D(X+)}} \mid \ell n\left(\frac{\lambda}{\sqrt{D(X+)}}\right)\mid\right) \quad , \quad \lambda \to 0+$$

(3.62)

$$\text{(b)} \quad d_-(\lambda) = O\left(\frac{\lambda}{\sqrt{D(X-)}} \mid \ell n\left(\frac{\lambda}{\sqrt{D(X-)}}\right)\mid\right) \quad , \quad \lambda \to 0+$$

where $d_+(\lambda)(d_-(\lambda))$ denotes the width of that part of the layer which is right (left) of X.

For the case (A) we obtain

$$\text{(a)} \quad d_+(\lambda) = O\left(\frac{\lambda}{\sqrt{\bar{n}(X+)+\tilde{p}(X+)}} \left(\sqrt{\bar{\psi}(X+)-\bar{\psi}(X-)} + \mid \ell n\left(\frac{\lambda}{\sqrt{\bar{n}(X+)+\bar{p}(X+)}}\right)\mid\right)\right)$$

(3.63)

$$\text{(b)} \quad d_-(\lambda) = O\left(\frac{\lambda}{\sqrt{n(X-)+p(X-)}} \left(\sqrt{\bar{\psi}(X+)-\bar{\psi}(X-)} + \mid \ell n\left(\frac{\lambda}{\sqrt{\bar{n}(X-)+\bar{p}(X-)}}\right)\mid\right)\right) \quad .$$

If the low injection condition (3.41) holds Theorem 3.2 gives a (physically relevant) solution of the reduced problem and (3.30), (3.63) give

$$\text{(a)} \quad d_+(\lambda) = O\left(\frac{\lambda}{\sqrt{D(X+)}} \left(\sqrt{\mid\frac{U}{U_T} + \frac{U_{bi}(\lambda)}{U_T}\mid} + \mid \ell n \frac{\lambda}{\sqrt{D(X+)}}\mid\right)\right)$$

(3.64)

$$\text{(b)} \quad d_-(\lambda) = O\left(\frac{\lambda}{\sqrt{\mid D(X-)\mid}} \left(\sqrt{\mid\frac{U}{U_T} + \frac{U_{bi}(\lambda)}{U_T}\mid} + \mid \ell n \frac{\lambda}{\sqrt{\mid D(X-)\mid}}\mid\right)\right) \quad .$$

These asymptotics are uniform as $U \to -\infty$.

4. Existence of Solutions

In this section we prove the existence of solutions of the singularly perturbed problem (2.11), (2.12) using the asymptotic expansions (3.2).

At first we derive an a priori estimate on the number of carrier pairs valid for the cases (A) and (B).

<u>Theorem 4.1.</u> Every solution of (2.11), (2.12) which satisfies (2.16), (2.17) and for which J_n, J_p do not change sign in $[-1,1]$ fulfills

$$(4.1) \qquad U > 0 \Longleftrightarrow J_n > 0, \; J_p > 0; \; J_n \not\equiv 0, \; J_p \not\equiv 0 \quad \text{on} \quad [-1,1]$$

$$(4.2) \qquad U = 0 \Longleftrightarrow J_n \equiv J_p \equiv 0$$

$$(4.3) \qquad U < 0 \Longleftrightarrow J_n < 0, \; J_p < 0; \; J_n \not\equiv 0, \; J_p \not\equiv 0 \quad \text{on} \quad [-1,1]$$

$$(4.4) \qquad \gamma^4 \lambda^4 e^{-\frac{|U|}{U_T}} \leq n(x)p(x) \leq \gamma^4 \lambda^4 e^{\frac{|U|}{U_T}}, \; x \in [-1,1] \quad .$$

The proof is completely analogous to the proof of Theorem 4.1 in Markowich, Ringhofer, Selberherr and Langer (1982) and requires only the equilibrium condition on the scaled recombination rates S_n, S_p. For $U = 0$ the current densities J_n, J_p vanish and the device is in thermal equilibrium. The np-product is constant $\gamma^4 \lambda^4$ throughout the device.

The estimate (4.4), the equilibrium condition and the continuity of S_n, S_p imply that S_n, S_p are small along a solution when $\frac{|U|}{U_T}$ is small. In particular, for the SRH-recombination rate (given by (2.14))

$$(4.5) \qquad |S_n| = |S_p| \leq \frac{\gamma^2 \lambda^2}{2\beta} (e^{\frac{|U|}{U_T}} - 1)$$

holds along every solution of (2.11), (2.12). Therefore it is intriguing to set

$$(4.6) \qquad S_n \equiv S_p \equiv 0$$

for sufficiently small $\frac{|U|}{U_T}$.

We now give existence proofs for (2.11), (2.12) in this case.

For the simple case (B) we show

-30-

Theorem 4.2. Assume that D fulfills the assumptions of Theorem 3.1 and that (4.6) holds. Then, if $0 < \lambda < \lambda_0$ and $|\frac{U}{U_T}| < \rho$ for ρ sufficiently small but independent of λ holds, the problem (2.11), (2.18), (2.19), (2.26), (2.27) has a solution which fulfills (2.16), (2.17) and

$$(4.7) \qquad \psi(x,\lambda) = \overline{\psi}(x) + \hat{\psi}(\frac{x-X}{\lambda}) + O(\lambda)$$

$$(4.8) \qquad n(x,\lambda) = D(x) + \hat{n}(\frac{x-X}{\lambda}) + O(\lambda)$$

$$(4.9) \qquad p(x,\lambda) = O(\lambda) \ (\geqslant 0)$$

$$(4.10) \qquad J_n(x,\lambda) = \overline{J}_n(x) + O(\lambda)$$

$$(4.11) \qquad J_p(x,\lambda) = O(\lambda)$$

uniformly on $[-1,1]$ where $\overline{\psi}, \overline{J}_n$ are given by (3.28)(e), (d), $\hat{\psi}$ is as in Theorem 3.3, case B and $\hat{n}$ fulfills (3.15).

Proof. The right hand sides of (4.7) – (4.11) are the sum of the reduced solutions as given in Theorem 3.1, the layer terms as of Theorem 3.3 ($\hat{p} \equiv 0$ holds since $\overline{p} \equiv 0$) and remainder terms. We denote these remainders by E_ψ, E_n, E_p, E_{J_n}, E_{J_p}. Inserting into (2.11) (with $s_n \equiv s_p \equiv 0$), using (3.23) and (3.6), (3.7) gives

$$(a) \qquad \lambda^2 E_\psi'' = E_n - E_p - \lambda^2 \overline{\psi}''$$

$$(b) \qquad E_n' = (\overline{\psi}+\hat{\psi})' E_n + (\overline{n}+\hat{n})E_\psi' + E_{J_n} + E_n E_\psi' + \phi_1(x,\lambda)$$

$$(4.12)(c) \qquad E_p' = -(\overline{\psi}+\hat{\psi})' E_p - E_{J_p} - E_p E_\psi' + \phi_2(x,\lambda)$$

$$(d) \qquad E_{J_n}' = 0$$

$$(e) \qquad E_{J_p}' = 0$$

where the functions ϕ_1, ϕ_2 satisfy

$$(4.13) \qquad \int_{-1}^{1} |(\phi_i(s,\lambda)| ds = O(\lambda), \ i = 1,2 \ .$$

Inserting into the boundary conditions (2.18), (2.19), (2.26), (2.27) shows that the boundary values for E_ψ, E_n, E_p at $x = \pm 1$ are $O(\lambda)$.

We define the operators:

-31-

$$\text{(a)} \quad (H_n g)(x) = \int_{-1}^{x} \exp(\overline{\psi}(x) + \hat{\psi}(\sigma(x)) - \overline{\psi}(s) - \hat{\psi}(\sigma(s))) g(s) ds$$

(4.13)

$$\text{(b)} \quad (H_p g)(x) = \int_{-1}^{x} \exp(-\overline{\psi}(x) - \hat{\psi}(\sigma(x)) + \overline{\psi}(s) + \hat{\psi}(\sigma(s))) g(s) ds$$

where $\sigma(x) = \dfrac{x-X}{\lambda}$ and rewrite (4.12) (b), (c) as integral equations

$$E_n = \exp(\overline{\psi}(x) + \hat{\psi}(\sigma(x)) - \overline{\psi}(-1) - \hat{\psi}(\sigma(-1))) E_n(-1) +$$

(4.14)(a)
$$(H_n(\overline{n}+\hat{n}) E_{\psi}')(x) + (H_n E_{J_n})(x) + (H_n E_n E_{\psi}')(x)$$

$$+ (H_n \phi_1)(x)$$

$$E_p = \exp(-\overline{\psi}(x) - \hat{\psi}(\sigma(x)) + \overline{\psi}(\sigma(-1))) E_p(-1) -$$

(4.14)(b)
$$- (H_p E_{J_p})(x) - (H_p E_p E_{\psi}')(x) + (H_p \phi_2)(x) \quad .$$

From (4.12)(c), (d) we get

(4.15)(a) $$E_{J_n} \equiv \text{const.} \quad \text{on} \quad [-1,1]$$

(4.15)(b) $$E_{J_p} \equiv \text{const.} \quad \text{on} \quad [-1,1]$$

since $E_{J_n}, E_{J_p} \in C([-1,1])$.

Because of (4.13) and since $|E_n(-1)| = 0(\lambda)$, $|E_p(-1)| = 0(\lambda)$ we obtain from (4.14)

$$\text{(a)} \quad E_n = H_n(\overline{n}+\hat{n}) E_{\psi}' + E_{J_n} H_n 1 + H_n E_n E_{\psi}' + 0(\lambda)$$

(4.16)

$$\text{(b)} \quad E_p = -E_{J_p} H_p 1 + H_p E_p E_{\psi}' + 0(\lambda) \quad .$$

Partial integration and (3.23) give

(4.17) $$(H_n(\overline{n}+\hat{n}) E_{\psi}')(x) = (\overline{n}+\hat{n}) E_{\psi} - (H_n \overline{J}_n E_{\psi})(x) + \lambda G_{n,\lambda} E_{\psi}$$

where $G_{n,\psi} : C([-1,1]) \to C([-1,1])$ is uniformly bounded (in λ). The continuity of $\overline{n} + \hat{n}$ at $x = X$ was used for the derivation of (4.17).

From (4.12)(b) we derive, after partial integration

(4.18) $$H_n E_n E_{\psi}' = E_n E_{\psi} - H_n \overline{n} E_{\psi}' E_{\psi} - H_n E_{J_n} E_{\psi} - H_n E_n E_{\psi}' E_{\psi} + \lambda F_{n,\lambda} E_{\psi}' E_{\psi} + 0(\lambda)$$

where $F_{n,\lambda} : C([-1,1]) \to C([-1,1])$ is uniformly bounded. Another partial integration gives

$$(4.19) \qquad H_n \bar{n} E_\psi' E_\psi = \frac{1}{2} \bar{n} E_\psi^2 - e^{\bar{\psi}} \int_{-1}^x (e^{-\bar{\psi}} \bar{n})' E_\psi^2(s)\, ds + 0(\lambda) \quad .$$

Combining (4.16) - (4.19) gives

$$(4.20)(a) \qquad E_n = (\bar{n}+\hat{n}) E_\psi - H_n \bar{J}_n E_\psi + E_{J_n} H_n 1 + \Lambda_{n,\lambda}(E_\psi, E_\psi', E_n, E_{J_n})$$

and proceeding analogously for E_p

$$(4.20)(b) \qquad E_p = -E_p H_p 1 + \Lambda_{p,\lambda}(E_\psi, E_\psi', E_p, E_{J_p})$$

where $\Lambda_{n,\lambda}$, $\Lambda_{p,\lambda}$ are nonlinear operators from $(C([-1,1]))^4$ into $C([-1,1])$ which fulfill

$$|\Lambda_{q,\lambda}(E_\psi, E_\psi', E_q, E_{J_q})|_{[-1,1]} \leq C_1(0(\lambda) + |E_\psi|_{[-1,1]}(\lambda + \lambda|E_\psi'|_{[-1,1]}$$

$$(4.21)(a) \qquad \qquad + |E_\psi|_{[-1,1]} + |E_{J_q}|_{[-1,1]} + |E_q|_{[-1,1]}|E_\psi'|_{[-1,1]}))$$

$$|D\Lambda_{q,\lambda}(E_\psi, E_\psi', E_q, E_{J_q})| \leq C_2(\lambda + |E_\psi|_{[-1,1]}$$

$$\qquad \qquad + |E_q|_{[-1,1]} + |E_{J_q}|_{[-1,1]}$$

$$(4.21)(b) \qquad \qquad + |E_q|_{[-1,1]}|E_\psi'|_{[-1,1]} + |E_q|_{[-1,1]}|E_\psi'|_{[-1,1]}$$

$$\qquad \qquad + |E_\psi'|_{[-1,1]}|E_\psi|_{[-1,1]} + \lambda|E_\psi'|_{[-1,1]})$$

where C_1, C_2 are independent of λ, "D" denotes the Fréchet derivative and $q = n,p$. The constants J_n, J_p can be determined from the boundary conditions for E_n, E_p at $x = +1$:

$$(4.22)(a) \qquad J_n \equiv \frac{(H_n \bar{J}_n E_\psi)(1) - \Lambda_{n,\lambda}(E_\psi, E_\psi', E_n, E_{J_n})(1)}{(H_n 1)(1)}$$

$$(4.22)(b) \qquad J_p \equiv \frac{\Lambda_{p,\lambda}(E_\psi, E_\psi', E_n, E_{J_n})(1)}{(H_p 1)(1)} \quad .$$

We remark that $(H_n 1)(1)$, $(H_p 1)(1)$ are bounded away from zero uniformly in λ.

We regard E_ψ, E_n, E_p, E_{J_n}, E_{J_p} as dwelling the space $A_x = C^1([-1,1]) \cap C_x^2([-1,1])$ which is equipped with the norm $|f|_\lambda = |f|_{[-1,1]} + \lambda^2 |f''|_{[-1,X) \cup (X,1]}$. (4.21)

-33-

implies that the mappings defined by the right hand sides of (4.20) are contractions in spheres of radius $0(\lambda)$ (centered at 0) and therefore (4.20)(a), (b) can be resolved with respect to E_n, E_p resp.:

(4.23)(a)
$$E_n = (\bar{n}+\hat{n})E_\psi - H_n\bar{J}_n E_\psi + (H_n\bar{J}_n E_\psi)(1)\frac{(H_n 1)(x)}{(H_n 1)(1)}$$
$$+ \Omega_{n,\lambda}(E_\psi, E'_\psi)$$

(4.23)(b)
$$E_p = \Omega_{p,\lambda}(E_\psi, E'_\psi) \quad .$$

The operators $\Omega_{n,\lambda}$, $\Omega_{p,\lambda}$ fulfill the estimates (4.21) when $\|E_n\|_{[-1,1]}$, $\|E_p\|_{[-1,1]}$, $\|E_{J_n}\|_{[-1,1]}$, $\|E_{J_p}\|_{[-1,1]}$ are substituted by the radius of the sphere in which the contraction mapping theorem is applied.

Inserting (4.23) into (4.12)(a) gives

(4.24)
$$\lambda^2 E''_\psi - (\bar{n}+\hat{n})E_\psi = (H_n\bar{J}_n E_\psi)(1)\frac{(H_n 1)(x)}{(H_n 1)(1)} - (H_n\bar{J}_n E_\psi)(x)$$
$$+ \Omega_{n,\lambda}(E_\psi, E'_\psi) - \Omega_{p,\lambda}(E_\psi, E'_\psi)$$
$$- \lambda^2\bar{\psi}'', \quad -1 < x < X \quad \text{and} \quad X < x < 1$$

subject to $0(\lambda)$ boundary conditions for E_ψ at $x = \pm 1$.

Since $\bar{n}+\hat{n}$ is positive and continuous on $[-1,1]$ the boundary value problem

(4.25)
$$\lambda^2 y'' - (\bar{n}+\hat{n})y = f(x), \quad y(-1) = y_-, \quad y(1) = y_+$$

has a unique solution $y \in A_x$ for all $y_-, y_+ \in R$, $f \in C_x([-1,1])$ which fulfills

(4.26)
$$\|y\|_\lambda \leq \text{const}(\|f\|_{[-1,1]} + |y_-| + |y_+|) \quad .$$

Since all estimates so far are uniform for $|\frac{U}{U_T}| \leq \rho$ and since

(4.27)
$$\|H_n\bar{J}_n E_\psi\|_{[-1,1]} \leq \text{const } \rho\|E_\psi\|_{[-1,1]}$$

holds with const independent of λ and $\rho \in [-\rho_0,\rho_0]$ (see (3.28)(d)) the contraction mapping theorem with ρ sufficiently small assures the existence of a locally unique E_ψ with

(4.28)
$$\|E_\psi\|_{[-1,1]} = 0(\lambda) \quad .$$

The existence statement of Theorem 4.2 follows immediately. The positivity of $p = E_p$ follows by investigating the higher order terms of the expansions similar to the proof of Theorem 3.2.

$\square$

This proof does not carry over to the case A since then H_n, H_p are not uniformly bounded (in λ) anymore. At first we rewrite (2.11) (with $S_n \equiv S_p \equiv 0$) as a second order problem.

From (2.11)(d), (e) we get

(4.29) $$J_n \equiv \text{const}, \quad J_p \equiv \text{const} \quad \text{on} \quad [-1,1]$$

(2.11)(b),(c) give

(4.30)

(a) $\quad n = n(1)e^{\psi(x)-\psi(1)} + J_n e^{\psi(x)} \int_1^x e^{-\psi(s)} ds$

(b) $\quad p = p(1)e^{\psi(1)-\psi(x)} - J_p e^{-\psi(x)} \int_1^x e^{\psi(s)} ds \quad .$

J_n, J_p have to be determined from the boundary conditions for n, p at $x = -1$

(4.31)(a) $$J_n \equiv \frac{\gamma^2\lambda^2\left(e^{-\frac{U_C}{U_T}} - e^{-\frac{U_A}{U_T}}\right)}{\int_{-1}^1 e^{-\psi(s)} ds}$$

(4.31)(b) $$J_p \equiv \frac{\gamma^2\lambda^2\left(e^{\frac{U_A}{U_T}} - e^{\frac{U_C}{U_T}}\right)}{\int_{-1}^1 e^{\psi(s)} ds}$$

(4.30), (4.31) immediately give $n > 0$, $p > 0$.

Without loss of generality we set $U_A = -U_C = \frac{U}{2}$ and obtain by inserting (4.30), (4.31) into (2.11)(a)

(4.32) $$\lambda^2\psi'' = 2\gamma^2\lambda^2\left(\sinh(\psi + \frac{U}{2U_T}) + \sinh(\frac{U}{2U_T})\left(e^{\psi}\frac{\int_1^x e^{-\psi(s)} ds}{\int_{-1}^1 e^{-\psi(s)} ds}\right.\right.$$

$$\left.\left. + e^{-\psi}\frac{\int_1^x e^{\psi(s)} ds}{\int_{-1}^1 e^{\psi(s)} ds}\right)\right) - D(x)$$

for $x \in [-1, X) \cup (X, 1]$ subject to the boundary conditions (2.20), (2.21) and the interface conditions (2.16).

At first we derive an asymptotic representation of the voltage current characteristic.

Lemma 4.1. Let $S_n \equiv S_p \equiv 0$ hold and assume that U fulfills the low injection condition (3.41), that D fulfills the assumptions of Theorem 3.2 (Case (A)) and

(4.33)(a)
$$\psi(x, \lambda) = \bar{\psi}(x) + \hat{\psi}(\frac{x-X}{\lambda}) + \theta(\lambda)$$

where the reduced solution $\bar{\psi}$ fulfills (3.30)(a) and $\hat{\psi}$ is given by Theorem 3.3. Then (3.31) implies

(4.33)(b)
$$J_n = \frac{\gamma^4 \lambda^4 (e^{\frac{U}{U_T}} - 1)}{\int_{-1}^{X} |D(s)| ds} \; (1 + O(|\theta(\lambda)| + d_+(\lambda) + d_-(\lambda) + \gamma^4 \lambda^4 e^{\frac{U}{U_T}}))$$

(4.33)(c)
$$J_p = \frac{\gamma^4 \lambda^4 (e^{\frac{U}{U_T}} - 1)}{\int_{X}^{1} D(s) ds} \; (1 + O(|\theta(\lambda)| + d_+(\lambda) + d_-(\lambda) + \gamma^4 \lambda^4 e^{\frac{U}{U_T}})) \; .$$

Proof. We obtain

$$e^{\psi_-} \int_{-1}^{1} e^{-\psi(s, \lambda)} ds = \int_{-1}^{X} \frac{D(s)}{D(-1)} ds + \int_{-1}^{X} \frac{D(s)}{D(-1)} e^{-\hat{\psi}(\frac{s-x}{\lambda})} O(|\theta(\lambda)|) ds$$

$$+ \int_{-1}^{X} \frac{D(s)}{D(-1)} (e^{-\hat{\psi}(\frac{s-x}{\lambda})} - 1) ds + e^{\psi_- - \psi_+} \int_{X}^{1} \frac{D(1)}{D(s)} ds$$

$$+ e^{\psi_- - \psi_+} \int_{X}^{1} \frac{D(1)}{D(s)} (e^{-\hat{\psi}(\frac{s-x}{\lambda})} - 1) ds$$

$$+ e^{\psi_- - \psi_+} \int_{X}^{1} \frac{D(1)}{D(s)} e^{-\hat{\psi}(\frac{s-x}{\lambda})} O(|\theta(\lambda)|) ds \; .$$

Using the estimates (3.43) and the layer widths given by (3.64) we derive

-36-

$$e^{\psi_-} \int_{-1}^{1} e^{-\psi(s,\lambda)} ds = \int_{-1}^{x} \frac{D(s)}{D(-1)} ds + 0(|\theta(\lambda)|) + 0(d_-(\lambda))$$

$$+ 0(\gamma^4 \lambda^4 e^{\frac{|U|}{U_T}}) + 0(d_+(\lambda)) \quad .$$

Therefore

$$J_n = \frac{e^{\psi_-} \gamma^2 \lambda^2 (e^{\frac{U}{2U_T}} - e^{-\frac{U}{2U_T}})}{e^{\psi_-} \int_{-1}^{1} e^{-\psi(s,\lambda)} ds} = \frac{\gamma^4 \lambda^4 (e^{\frac{U}{U_T}} - 1)}{e^{\psi_-} \int_{-1}^{1} e^{-\psi(s,\lambda)} ds (1 + 0(\gamma^4 \lambda^4))}$$

$$= \frac{\gamma^4 \lambda^4 (e^{\frac{U}{U_T}} - 1)}{\int_{-1}^{x} |D(s)| ds} (1 + 0(|\theta(\lambda)| + d_+(\lambda) + d_-(\lambda) + \gamma^4 \lambda^4 e^{\frac{U}{U_T}}))$$

follows. The proof for the asymptotic representation of J_p is analogous.

If $D_n \approx D_p$ holds (which implies that J_n, J_p have the same scaling factors) and if a solution of (4.32) subject to the boundary conditions $\psi(-1,\lambda) = \psi_-$, $\psi(1,\lambda) = \psi_+$ exists for which $\theta(\lambda) \to 0$ as $\lambda \to 0+$, then the total voltage current characteristic of the pn-function is given by

$$(4.33)(d) \qquad J \equiv J_n + J_p \equiv \gamma^4 \lambda^4 \left(\frac{1}{\int_{-1}^{x} |D(s)| ds} + \frac{1}{\int_{x}^{1} D(s) ds} \right)(e^{\frac{U}{U_T}} - 1)(1 + o(1)) \quad .$$

The same asymptotic form of J can be found in Sze (1969) (and other standard books on semiconductor physics), however the derivation used there heavily relies on physical arguments.

We now prove an existence Theorem for the case (A) under a slightly sharper assumption on $\left| \frac{U}{U_T} \right|$ then that used in Theorem 4.2.

Theorem 4.3. Let D fulfill the assumption of Theorem 3.2 and $D \in C_x^2([-1,1])$. Assume that

$$\left| \frac{U}{U_T} \right| < \frac{\rho}{|\ln \gamma^2 \lambda^2|}$$

holds for some $\rho > 0$ sufficiently small but independent of λ. Then there is a solution $\psi(x,\lambda)$ of (4.32) subject to (2.20), (2.21), (2.16) and ψ fulfills

-37-

(4.34)
$$\psi(x,\lambda) = \bar{\psi}(x) + \hat{\psi}(\frac{x-X}{\lambda}) + O(d_+(\lambda) + d_-(\lambda) + (\gamma^4\lambda^4)^\gamma)$$

for some $\gamma > 0$ where $\bar{\psi}$ is the reduced solution as given by Theorem 3.2 (with $S_n \equiv S_p \equiv 0$) and $\hat{\psi}$ is the layer solution given by Theorem 3.3.

<u>Proof</u>. We define

$$\psi_+^0 = \ell n\left[\frac{D(1) + \sqrt{D(1)^2 + 4\gamma^4\lambda^4}}{2\gamma^2\lambda^2}\right]$$

$$\psi_-^0 = \ell n\left[\frac{2\gamma^2\lambda^2}{-D(-1) + \sqrt{D(-1)^2 + 4\gamma^4\lambda^4}}\right] \quad.$$

Since $2\gamma^2\lambda^2 = \dfrac{D(1)}{\sinh \psi_+^0} = \dfrac{D(-1)}{\sinh \psi_-^0}$ holds, the problem (4.32), (2.20), (2.21) with $U = 0$

can be rewritten as

(4.35)(a)
$$\lambda^2\psi_0'' = D(1) \frac{\sinh \psi_0}{\sinh \psi_+^0} - D(x), \quad X < x < 1$$

(4.35)(b)
$$\lambda^2\psi_0'' = D(-1) \frac{\sinh \psi_0}{\sinh \psi_-^0} - D(x), \quad -1 < x < X$$

(4.35)(c)
$$\psi_0(1,\lambda) = \psi_+^0, \quad \psi_0(-1,\lambda) = \psi_-^0$$

(4.35)(d)
$$\psi_0 \in C^1([-1,1]) \quad.$$

We now regard ψ_+^0, ψ_-^0 as parameters independent of λ (as in Section 3).

Then the reduced solution $\bar{\psi}_0$ has the asymptotic form given by Theorem 3.2.

(4.36) $\bar{\psi}_0(x) = \begin{cases} \text{area } \sinh(\frac{D(x)}{D(1)} \sinh \psi_+^0) = \psi_+^0 + \ell n \frac{D(x)}{D(1)} + O(e^{-2\psi_+^0}), \quad X < x < 1 \\[4ex] \text{area } \sinh(\frac{D(x)}{D(-1)} \sinh \psi_-^0) = \psi_-^0 - \ell n \frac{D(x)}{D(-1)} + O(e^{-2\psi_-^0}), \quad -1 < x < X \end{cases}$

and the internal layer solution $\hat{\psi}_0$ is as in Theorem 3.3 but subject to the changed

-38-

interface condition $\overset{\bullet}{\psi}(0+) - \overset{\bullet}{\psi}(0-) = \lambda(\overline{\psi}'(X-)\overline{\psi}'(X+))$.

In order to investigate stability we substitute $u_0 = \dfrac{\psi_0}{\psi_+^0}$ in (4.32), (2.20), (2.21)

with $U = 0$ and obtain the problem

(4.37)(a) $\qquad \lambda^2 u_0'' = \dfrac{D(1)}{\psi_+^0} \dfrac{\sinh \psi_+^0 u_0}{\sinh \psi_+^0} - \dfrac{1}{\psi_+^0} D(x)$ $-1 < x < X, \; X < x < 1$

(4.37)(b) $\qquad u_0(-1) = \dfrac{\psi_-^0}{\psi_+^0}, \; u_0(1) = 1$

(4.37)(c) $\qquad u_0 \in C^1([-1,1])$.

We denote (4.37) by $F_0(u_0, \lambda, \psi_+^0, \psi_+^-)$, where $F_0(\cdot, \lambda, \psi_+^0, \psi_-^0) : A_X \to C_X([-1,1]) \times \mathbb{R}^2$ and

investigate the equation

(4.38) $\qquad\qquad L_{\lambda,0} v \equiv D_u F_0(\overline{u}_0 + \hat{u}_0, \lambda, \psi_+^0, \psi_-^0) v = (f, \alpha, \beta)$

where $\overline{u}_0 = \dfrac{\overline{\psi}_0}{\psi_+^0}, \; \hat{u}_0 = \dfrac{\hat{\psi}_0}{\psi_+^0}$. (4.38) is equivalent to

(4.39)(a) $\qquad \lambda^2 v'' - D(1) \dfrac{\sinh(\overline{\psi}_0 + \hat{\psi}_0)}{\sinh \psi_+^0} v = f(x), \; -1 < x < X, \; X < x < 1$

(4.39)(b) $\qquad v(-1) = \alpha, \; v(1) = \beta$

(4.39)(c) $\qquad v \in C^1([-1,1])$.

We remark that $\overline{\psi}_0 + \hat{\psi}_0(\frac{\cdot}{\lambda}) \in C^1([-1,1])$.

The maximum principle immediately implies uniqueness of the solution of (4.39) and the

Fredholms alternative gives existence. To get a bound for the inverse of (4.39)

we construct the barrier function:

(4.40) $\qquad\qquad v_b(x, \lambda) = K + \exp\left(- \dfrac{\beta(x-X)^2}{\lambda^2(\psi_+^0 - \psi_-^0)}\right)$

where the constants $K > 0, \; \beta > 0$ will be determined thereafter.

-39-

We compute

$$(4.41) \qquad (L_{\lambda,0} v_b)(x) = \left(\frac{2\beta}{\psi_+^0 - \psi_-^0} \left(\frac{2\beta(x-X)^2}{\lambda^2(\psi_+^0 - \psi_-^0)} - 1\right) \exp\left(-\frac{\beta(x-X)^2}{\lambda^2(\psi_+^0 - \psi_-^0)}\right) - \right.$$

$$D(1) \frac{\cosh(\bar{\psi}_0 + \hat{\psi}_0)}{\sinh \psi_+^0} \left(K + \exp\left(-\frac{\beta(x-X)^2}{\lambda^2(\psi_+^0 - \psi_-^0)}\right)\right) ,$$

$$\left. K + \exp\left(-\frac{\beta(1+X)^2}{\lambda^2(\psi_+^0 - \psi_-^0)}\right), \; K + \exp\left(-\frac{(\beta(1-X)^2}{\lambda^2(\psi_+^0 - \psi_-^0)}\right)\right) .$$

We denote the first component of $(L_{\lambda,0} v_b)(x)$ by $(L_{\lambda,0}^{(1)} v_h)(x)$ and get

$$(L_{\lambda,0}^{(1)} v_b)(x) \leq - \frac{\beta}{\psi_+^0 - \psi_-^0} e^{-\frac{1}{2}} , \quad x \in \left[X - \frac{\lambda\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}} , \; X + \frac{\lambda\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}}\right]$$

and

$$\left(\frac{2\beta}{\psi_+^0 - \psi_-^0} \left(\frac{2\beta(x-X)^2}{\lambda^2(\psi_+^0 - \psi_-^0)} - 1\right) \exp\left(-\frac{\beta(x-X)^2}{\lambda^2(\psi_+^0 - \psi_-^0)}\right)\right)^+ \leq \frac{4\beta e^{\frac{3}{2}}}{\psi_+^0 - \psi_-^0} , \quad x \in [-1,1]$$

where $(f)^+$ denotes the positive part of the function f.

Now we use the estimates for $\hat{\psi}$ given in Theorem 3.3 for fixed $0 < \delta < 1$. At first

we estimate

$$D(1) \frac{\cosh(\bar{\psi}_0 + \hat{\psi}_0)}{\sinh \psi_+^0} \geq \begin{cases} D(x) e^{\hat{\psi}_0}, \; x \, C-[X,1] \\ \\ |D(x)| e^{-\hat{\psi}_0}, \; x \, C-[-1,X] \end{cases} .$$

Since $\hat{\psi}_0$ is monotonically increasing on $[-1,X)$ and on $(X,1]$ this yields

$$\left(L_{\lambda,0}^{(1)}\right)(x) \leq \frac{4\beta e^{\frac{3}{2}}}{\psi_+^0 - \psi_-^0} - K \begin{cases} D(x)\, \exp\!\left(\hat{\psi}_0\!\left(\dfrac{\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}}\right)\right), & x \in \left[X + \dfrac{\lambda\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}},\ 1\right] \\[2em] |D(x)|\exp\!\left(-\hat{\psi}_0\!\left(-\dfrac{\sqrt{\psi_+^0\psi_-^0}}{2\sqrt{\beta}}\right)\right), & x \in \left[-1, X - \dfrac{\lambda\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}}\right] \end{cases}$$

(note that $\sigma = \frac{x-X}{\lambda}$). We now choose β

$$\sqrt{\beta} = \frac{1}{2}\min\left(\frac{1}{2E_\delta},\ \frac{(1-\delta)\sqrt{\bar{n}(X+)+\bar{p}(X+)}}{2D_\delta},\ \frac{(1-\delta)\sqrt{\bar{n}(X-)+\bar{p}(X-)}}{2D_\delta}\right)$$

such that

$$\left(L_{\lambda,0}^{(1)} v_b\right)(x) \leq \frac{4\beta e^{\frac{3}{2}}}{\psi_+^0 - \psi_-^0} - Ke^{-1} \begin{cases} D(x)\ , & x \in \left[X + \dfrac{\lambda\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}},\ 1\right] \\[2em] |D(x)|\ , & x \in \left[-1, X - \dfrac{2\sqrt{\psi_+^0 - \psi_-^0}}{2\sqrt{\beta}}\right] \end{cases}$$

holds and

$$K = \frac{1 + \dfrac{4\beta e^{\frac{3}{2}}}{\psi_+^0 - \psi_-^0}}{e^{-1} \min_{x\in[-1,1]} |D(x)|}\ .$$

Then

$$\left(L_{\lambda,0}^{(1)} v_b\right)(x) \leq -\min\left(1,\ \frac{\beta e^{-\frac{1}{2}}}{\psi_+^0 - \psi_-^0}\right)\ ,\quad x \in [-1,1]$$

-41-

holds. The maximum principle implies the estimate

$$\|L_{\lambda,0}^{-1}\|_{C_X([-1,1])\times \mathbb{R}^2 \to C([-1,1])} \leq \text{const.}\ (\psi_+^0 - \psi_-^0)\ .$$

(4.39)(a) gives

(4.42)
$$\|L_{\lambda,0}^{-1}\|_{C_X([-1,1])\times \mathbb{R}^2 \to A_X} \leq \text{const.}\ (\psi_+^0 - \psi_-^0)\ .$$

The Fréchet-derivative $D_u F_0$ is locally Lipschitz continuous:

$$\|D_u F_0(w_1,\lambda,\psi_+^0,\psi_-^0) - D_u F_0(w_2,\lambda,\psi_+^0,\psi_-^0)\|_{A_X \to C_X([-1,1])\times \mathbb{R}^2} \leq$$

(4.43)
$$\text{const.}\ \psi_+^0 \|w_1 - w_2\|_{A_X}$$

for w_1, w_2 in a sphere centered at $\bar{u}_0 + \hat{u}_0 = \dfrac{\bar{\psi}_0 + \hat{\psi}_0}{\psi_+^0}$ with radius $\dfrac{\text{const}}{\psi_+^0}$.

Now we rewrite (4.32) as

(4.44)
$$\lambda^2 u'' = \frac{D(1)}{\psi_+^0} \frac{\sinh(\psi_+^0 u + \frac{U}{2U_T})}{\sinh \psi_+^0} + \frac{D(1)}{\psi_+^0} \sinh(\frac{U}{2U_T}) \frac{H(\psi_+^0 u)}{\sin h\ \psi_+^0}$$

$$- \frac{D(x)}{\psi_+^0}\ ,\quad x \in [-1,X)0(X,1]$$

where $H(\psi)$ denotes the integral operator on the right hand side of (4.32). We denote

(4.44) subject to the boundary conditions $u(-1) = \dfrac{\psi_-}{\psi_+^0}$, $u(1) = \dfrac{\psi_+}{\psi_+^0}$ (with $\psi_- = \psi_-^0 + \dfrac{U}{2}$,

$\psi_+ = \psi_+^0 - \dfrac{U}{2}$) by $F_U(u,\lambda,\psi_+^0,\psi_-^0) = 0$. The Fréchet derivative $L_{\lambda,U} = D_u F_U(\bar{\psi}_0 + \hat{\psi}_0,\lambda,\psi_+^0,\psi_-^0)$ is given by

(4.45)
$$(L_{\lambda,U} v)(x) = (\lambda^2 v'' - \frac{D(1)\sinh(\bar{\psi}_0 + \hat{\psi}_0 + \frac{U}{2U_T})}{\sinh \psi_+^0} -$$

$$D(1) \frac{\sinh \frac{U}{2U_T}}{\sinh \psi_+^0} (D_u H(\bar{\psi}_0 + \hat{\psi}_0)v)(x),\ v(-1),\ v(1))\ .$$

-42-

where

$$|D_u H(\bar{\psi}_0 + \hat{\psi}_0)|_{C_X([-1,1]) \,\to\, C_X([-1,1])} \leq \text{const. sinh } \psi_+^0 \,.$$

Therefore, if

(4.46)
$$\left|\frac{U}{U_T}\right| < \frac{\rho}{\psi_+^0 - \psi_-^0}, \quad \rho \text{ suff. small}$$

holds, $L_{\lambda,U}^{-1}$ fulfills the estimate (4.42). Moreover $D_u F_U$ is locally Lipschitz continuous and fulfills (4.43).

Since $|\bar{\psi} + \hat{\psi} - (\bar{\psi}_0 - \hat{\psi}_0)|_{[-1,1]} \leq \text{const } |U|$ a simple perturbation argument shows that also $D_u F_U(\frac{\bar{\psi} + \hat{\psi}}{\psi_+^0}, \lambda, \psi_+^0, \psi_-^0)^{-1}$ fulfills the stability estimate (4.42) and that $D_u F_U$ is locally Lipschitz continuous around $\bar{\psi} + \hat{\psi}$ ((4.43) holds) if U is restricted by (4.46).

We now insert $\bar{u} + \hat{u} = \frac{\bar{\psi} + \hat{\psi}}{\psi_+^0}$ into F_u in order to compute the 'local error'. We obtain

$$F_U(\bar{u} + \hat{u}, \lambda, \psi_+^0, \psi_-^0) = (\frac{\lambda^2}{\psi_+^0} \bar{\psi}'' + \frac{1}{\psi_+^0} \hat{\psi}(\sigma) - [\frac{D(1)}{\psi_+^0} \frac{\sinh(\bar{\psi} + \hat{\psi} + \frac{U}{2U_T})}{\sinh \psi_+^0} +$$

$$\frac{D(1)}{\psi_+^0} \sinh(\frac{U}{2U_T}) \frac{H(\bar{\psi} + \hat{\psi})}{\sinh \psi_+^0} - \frac{D(x)}{\psi_+^0}], \frac{\hat{\psi}(\frac{-1-x}{\lambda})}{\psi_+^0}, \frac{\hat{\psi}(\frac{1-x}{\lambda})}{\psi_+^0})$$

with $\sigma = \frac{x-X}{\lambda}$. Obviously

$$\frac{D(1)}{\psi_+^0} \sinh(\frac{U}{2U_T}) \frac{H(\bar{\psi} + \hat{\psi})}{\sinh \psi_+^0} = \frac{1}{\psi_+^0} (\tilde{J}_n e^{\bar{\psi} + \hat{\psi}} \int_1^x e^{-\bar{\psi} - \hat{\psi}} ds + \tilde{J}_p e^{-\bar{\psi} - \hat{\psi}} \int_1^x e^{\bar{\psi} + \hat{\psi}} ds)$$

where $\tilde{J}_n$, $\tilde{J}_p$ are given by (4.31)(a), (b) when ψ is substituted by $\bar{\psi} + \hat{\psi}$.

We rewrite

$$\tilde{J}_p e^{-\bar{\psi} - \hat{\psi}} \int_1^x e^{\bar{\psi} + \hat{\psi}} ds = \tilde{J}_p e^{-\bar{\psi} - \hat{\psi}} \int_{-1}^x e^{\bar{\psi} + \hat{\psi}} - \frac{\sinh(\frac{U}{2U_T})}{\sinh \psi_+^0} e^{-\bar{\psi} - \hat{\psi}} \,.$$

Since $\hat{\psi}$ is positive on $[-1,X)$ and negative on $[X,1]$ and monotonically decreasing on $[-1,X)$ and on $(X,1]$ we get

-43-

$$\left.\begin{array}{l} |e^{\bar{\psi}+\hat{\psi}} \int_{1}^{x} e^{-\bar{\psi}-\hat{\psi}} ds| < const \\[2mm] |e^{-\bar{\psi}-\hat{\psi}} \int_{-1}^{x} e^{\bar{\psi}+\hat{\psi}} ds| < const \end{array}\right\} \quad x \in [-1,1]$$

Lemma (4.1) with $\theta(\lambda) = 0$ gives

$$\frac{D(1)}{\psi_{+}^{0}} \sinh(\frac{U}{2U_{T}}) \frac{H(\bar{\psi}+\hat{\psi})}{\sinh \psi_{+}^{0}} = 0\left(\frac{\gamma^{4}\lambda^{4}}{\psi_{+}^{0}} |e^{\frac{U}{U_{T}}} - 1|\right) - \frac{D(1)\sinh(\frac{U}{2U_{T}})}{\sinh \psi_{+}^{0}} e^{-\bar{\psi}-\hat{\psi}}$$

if the low injection condition holds.

We calculate, using $\dfrac{D(1)}{2 \sinh \psi_{+}^{0}} = \dfrac{D(-1)}{2 \sinh \psi_{-}^{0}}$:

$$\overset{..}{\hat{\psi}}(\sigma) - [D(1) \frac{\sinh(\bar{\psi}+\hat{\psi}+\frac{U}{2U_{T}})}{\sinh \psi_{+}^{0}} - D(1) \frac{\sinh(\frac{U}{2U_{T}})}{\sinh \psi_{+}^{0}} e^{-\bar{\psi}-\hat{\psi}} - D(x)]$$

$$= \overset{..}{\hat{\psi}}(\sigma) - [D(1) \frac{e^{\bar{\psi}+\hat{\psi}+\frac{U}{2U_{T}}}}{2 \sinh \psi_{+}^{0}} - D(1) \frac{e^{-\bar{\psi}-\hat{\psi}+\frac{U}{2U_{T}}}}{2 \sinh \psi_{+}^{0}} - D(x)]$$

(4.47)
$$= \left\{ \begin{array}{l} [D(x) \dfrac{e^{\psi_{+}^{0}}}{2 \sinh \psi_{+}^{0}} e^{\hat{\psi}} - \dfrac{D(1)^{2} e^{-\psi_{+}^{0}+\frac{U}{U_{T}}}}{2D(x)\sinh \psi_{+}^{0}} e^{-\hat{\psi}} + 0(\gamma^{4}\lambda^{4} e^{\frac{U}{U_{T}}}) e^{\hat{\psi}} \\[6mm] [\dfrac{D(-1)^{2} e^{\psi_{-}^{0}+\frac{U}{U_{T}}}}{2D(x)\sinh \psi_{-}^{0}} e^{\hat{\psi}} - D(x) \dfrac{e^{-\psi_{-}^{0}}}{2 \sinh \psi_{-}^{0}} e^{-\hat{\psi}} + 0(\gamma^{4}\lambda^{4} e^{\frac{U}{U_{T}}}) e^{\hat{\psi}} \end{array} \right.$$

$$+ 0(\gamma^{4}\lambda^{4} e^{\frac{U}{U_{T}}}) e^{-\hat{\psi}} - D(x)] \quad , \quad x \in (X,1)$$

$$+ 0(\gamma^{4}\lambda^{4} e^{\frac{U}{U_{T}}}) e^{-\hat{\psi}} - D(x)] \quad , \quad x \in [-1,X) \quad .$$

Using Theorem 3.2 gives

-44-

$$D(x) \frac{e^{\psi_+^0}}{2 \sinh \psi_+^0} = \bar{n}(X+) + (D(x) - D(X+)) + 0(\gamma^4\lambda^4(1 + e^{\frac{U}{U_T}}))$$

$$\frac{D(1)^2 e^{-\psi_+^0 + \frac{U}{U_T}}}{2D(x)\sinh \psi_+^0} = \bar{p}(X+) + 0(\gamma^4\lambda^4(1 + e^{\frac{U}{U_T}}))$$

$$\left. \vphantom{\frac{D(1)^2 e^{-\psi_+^0 + \frac{U}{U_T}}}{2D(x)\sinh \psi_+^0}} \right\} \quad x \in (X,1)$$

$$\frac{D(-1)^2 e^{\psi_-^0 + \frac{U}{U_T}}}{2D(x)\sinh \psi_-^0} = \bar{n}(X-) + 0(\gamma^4\lambda^4(1 + e^{\frac{U}{U_T}}))$$

$$\frac{D(x)}{2 \sinh \psi_-^0}^{-\psi_-^0} = \bar{p}(X-) + (D(X-) - D(x)) + 0(\gamma^4\lambda^4(1 + e^{\frac{U}{U_T}}))$$

$$\left. \vphantom{\frac{D(-1)^2 e^{\psi_-^0 + \frac{U}{U_T}}}{2D(x)\sinh \psi_-^0}} \right\} \quad x \in [-1,X)$$

Since $\hat{\psi}$ solves the layer-equations (3.23)(a),(b) we get for (4.47)

(4.48)

$$\left\{ \begin{array}{l} (D(x)-D(X+)(e^{\hat{\psi}}-1) + 0(\gamma^4\lambda^4(1 + e^{\frac{U}{U_T}}))e^{\hat{\psi}} + 0(\gamma^4\lambda^4(1 + e^{\frac{U}{U_T}}))e^{-\hat{\psi}}, \; x \in (X,1] \\[3em] (D(x)-D(X-))(e^{-\hat{\psi}}-1) + 0(\gamma^4\lambda^4(1 + e^{\frac{U}{U_T}}))e^{\hat{\psi}} \\[2em] \qquad + 0(\gamma^4\lambda^4(1 + e^{\frac{U}{U_T}}))e^{-\hat{\psi}}, \; x \in [-1,X) \end{array} \right.$$

$\hat{\psi}$ is negative on $(X,1]$ and positive on $[-1,X)$ and since $D \in C_X^1([-1,1])$ we obtain

$$|(D(x)-D(X+))(e^{\hat{\psi}}-1)| \leq \text{const. } d_+(\lambda), \; x \in (X,1]$$

$$|(D(x)-D(X-))(e^{-\hat{\psi}}-1)| \leq \text{const. } d_-(\lambda), \; x \in [-1,X) \quad .$$

Also, (3.50)(a), (b) imply

-45-

$$e^{-\hat{\psi}(\frac{x-X}{\lambda})} \leqslant e^{-\hat{\psi}(0+)} \leqslant \text{const. } \exp\left(\frac{|D(X-)|}{D(X+)-D(X-)}\ (\psi_+ - \psi_-)\right)$$

$$\leqslant \text{const. } \exp\left(\frac{|D(X-)|}{D(X+)-D(X-)}\ \frac{U}{U_T}\right)(\gamma^4 \lambda^4)^{\frac{-|D(X-)|}{D(X+)-D(X-)}}, \quad x \in (X,1]$$

and analogously

$$e^{\hat{\psi}(\frac{x-X}{\lambda})} \leqslant e^{\hat{\psi}(0-)} \leqslant \text{const. } \exp\left(\frac{-D(X+)}{D(X+)-D(X-)}\ \frac{U}{U_T}\right)(\gamma^4 \lambda^4)^{\frac{-D(X+)}{D(X+)-D(X-)}}, \quad x \in [-1,X) \quad .$$

Therefore the expansion (4.48) is bounded by const $(d_-(\lambda) + d_+(\lambda) + \exp(\alpha|\frac{U}{U_T}|)(\gamma^4 \lambda^4)^\gamma)$

where $\alpha > 0$, $\gamma > 0$ holds.

These estimates and $\bar{\psi} \in C_X^2([-1,1])$ for $D \in C_X^2([-1,1])$ imply that

$$(4.49) \qquad \|F_U(\bar{u}+\hat{u},\lambda,\psi_+^0,\psi_-^0)\|_{C_X([-1,1])} \leqslant \text{const } \frac{d_-(\lambda) + d_+(\lambda) + e^{\alpha|\frac{U}{U_T}|}(\gamma^4 \lambda^4)^\gamma}{\psi_+^0}$$

holds.

The stability estimate (4.46) (which holds for $L_{\lambda,U}^{-1}$ if U fulfills (4.46)), Lipschitz continuity of $D_u F_U$ and (4.49) make it possible to apply the version of the implicit function theorem given by Spijker (1972), which implies that $F_U(u,\lambda,\psi_+^0,\psi_-^0) = 0$ has a solution u^* which is unique in a sphere in A_X with radius $\frac{x}{(\psi_+^0)^2}$ for sufficiently small x centered at $u+\hat{u}$ and the estimate

$$\|u-u^*\|_{A_X} \leqslant \text{const.}(d_+(\lambda) + d_-(\lambda) + (\gamma^4 \lambda^4)^\beta)$$

holds for $|\frac{U}{U_T}| \leqslant \frac{\rho}{\psi_+^0}$ where ρ is sufficiently small but independent of λ.

$\square$

We remark that the size reduction on $|\frac{U}{U_T}|$ comes from the interpretation of (4.32) as perturbation of the equilibrium problem (U=0) which was heavily used for the stability proof. The numerical results demonstrated in the next Section indicate that existence and validity of the asymptotic expansions hold under much weaker restrictions on $|\frac{U}{U_T}|$.

5. Numerical Experiments

We demonstrate numerical results for two pn-junctions in the high injection case, that is $U \gtrsim U_{bi}(\lambda)$. The existence Theorem 3.2 for the reduced problem does not hold if the low injection condition (3.41) is neglected. Both functions we investigate have characteristic length $2\ell = 5 \times 10^{-3}$ cm, the doping profile of the first pn-junction (called junction I in the sequel) is

$$(5.1) \qquad C(z) = \begin{cases} -0.5 \times 10^{17} \text{cm}^3 & , \quad -\ell \leqslant z < \frac{\ell}{2} \\ 10^{17} \text{cm}^3 & , \quad \frac{\ell}{2} < z \leqslant \ell \end{cases}$$

and for the second junction (called junction II in the sequel)

$$(5.2) \qquad C(z) = \begin{cases} -10^{15} \text{cm}^3 & , \quad -\ell \leqslant z < \frac{\ell}{2} \\ 10^{17} \text{cm}^3 & , \quad \frac{\ell}{2} < z \leqslant \ell \ . \end{cases}$$

Accurately speaking, both devices are pn^+ junctions. In both cases $\bar{C} = 10^{17}$cm^3. This and the numerical values for the parameters from Table 1 gives for both devices using the formulae (2.6), (2.9):

$$(5.3) \qquad \lambda^2 = 0.4 \times 10^{-6} \ , \quad \gamma^2 = 0.25 \ .$$

For junction I we obtain

$$(5.4) \qquad \frac{\min\limits_{z} |C(z)|}{\max\limits_{z} |C(z)|} = \frac{1}{2}$$

and for junction II

$$(5.5) \qquad \frac{\min\limits_{z} |C(z)|}{\max\limits_{z} |C(z)|} \approx 10^{-2} \ .$$

For both cases the singular perturbation approach seems applicable because (2.22), (2.23) 'holds' (the order of magnitude of λ^2 and $\min\limits_{z} |C(z)|/\max\limits_{z} |C(z)|$ as given by (5.3) and (5.4), (5.5) resp. are clearly different and $n_i = 10^{10}$ while $\bar{C} = 10^{17}$).

-47-

The built-in-voltage (calculated using (2.25)) for junction I is

(5.6) $$U_{bi} = -0.79V$$

and for junction II

(5.7) $$U_{bi} = -0.69V \quad .$$

All calculations described in the sequel were performed on the CDC-Cyber 74 computer of the Technical University of Vienna with the boundary-and-interface-problem solver PASVA4 written by M. Lentini and V. Pereyra. The SRH-recombination term was used. Figures 1-3 show the reduced solutions of a typical high injection case for junction I with U = 1.39V. The majority carrier densities (n on the n-side, which is the interval $(\frac{1}{2}, 1]$ and p on the p-side which is the interval $[-1, \frac{1}{2})$) are larger than the doping $|D|$ (except at the boundaries $x = \pm 1$).

The reduced solutions for a high-injection case (U = 0.99V) for junction II are shown in Figures 4-6.

Since in both cases the applied voltages are significantly larger than the absolute value of the built-in-voltage, the existence Theorem 3.2 for the reduced problem cannot be applied. However the presented numerical results give a strong indication for the existence of reduced solutions even in the high injection case.

Figure 7-9 and 10-12 show the solutions of the singularly perturbed (full) problem with $U = |U_{bi}|$ for junction I and II respectively. The internal layer in the components ψ, n, p is clearly visible. The solutions of the corresponding reduced problems (whose existence is also not covered by Theorem 3.2) were also computed and they agreed up to graphical accuracy with the full solutions away from the layer (see Figure 13, which shows the reduced solutions n, p for function I). In fact, the reduced solutions were used as starting guesses for the numerical method to compute the full solutions and convergence was achieved in a few steps.

This indicates that the asymptotic expansions are valid for a much larger range of U values than given in Theorem 4.2.
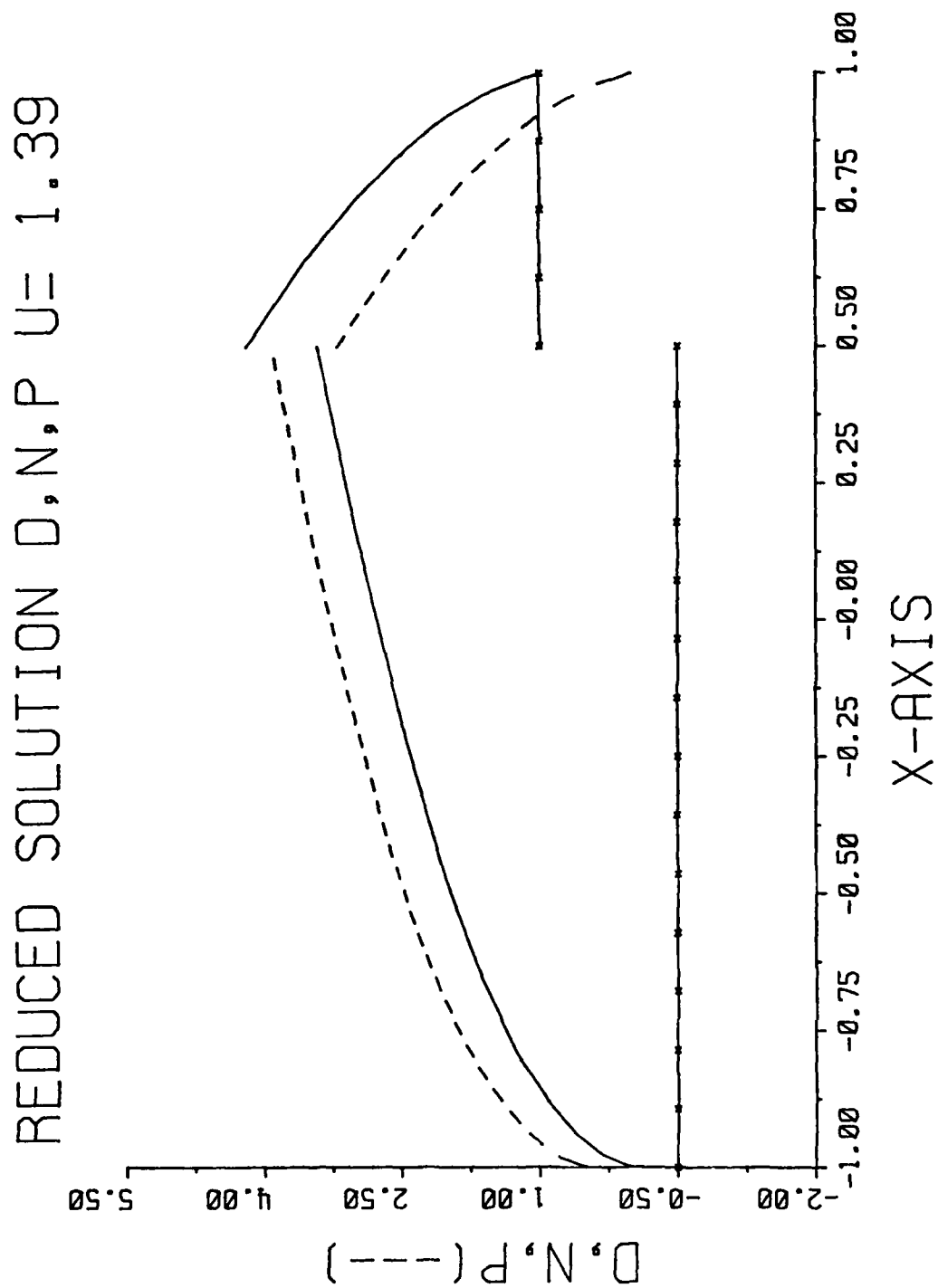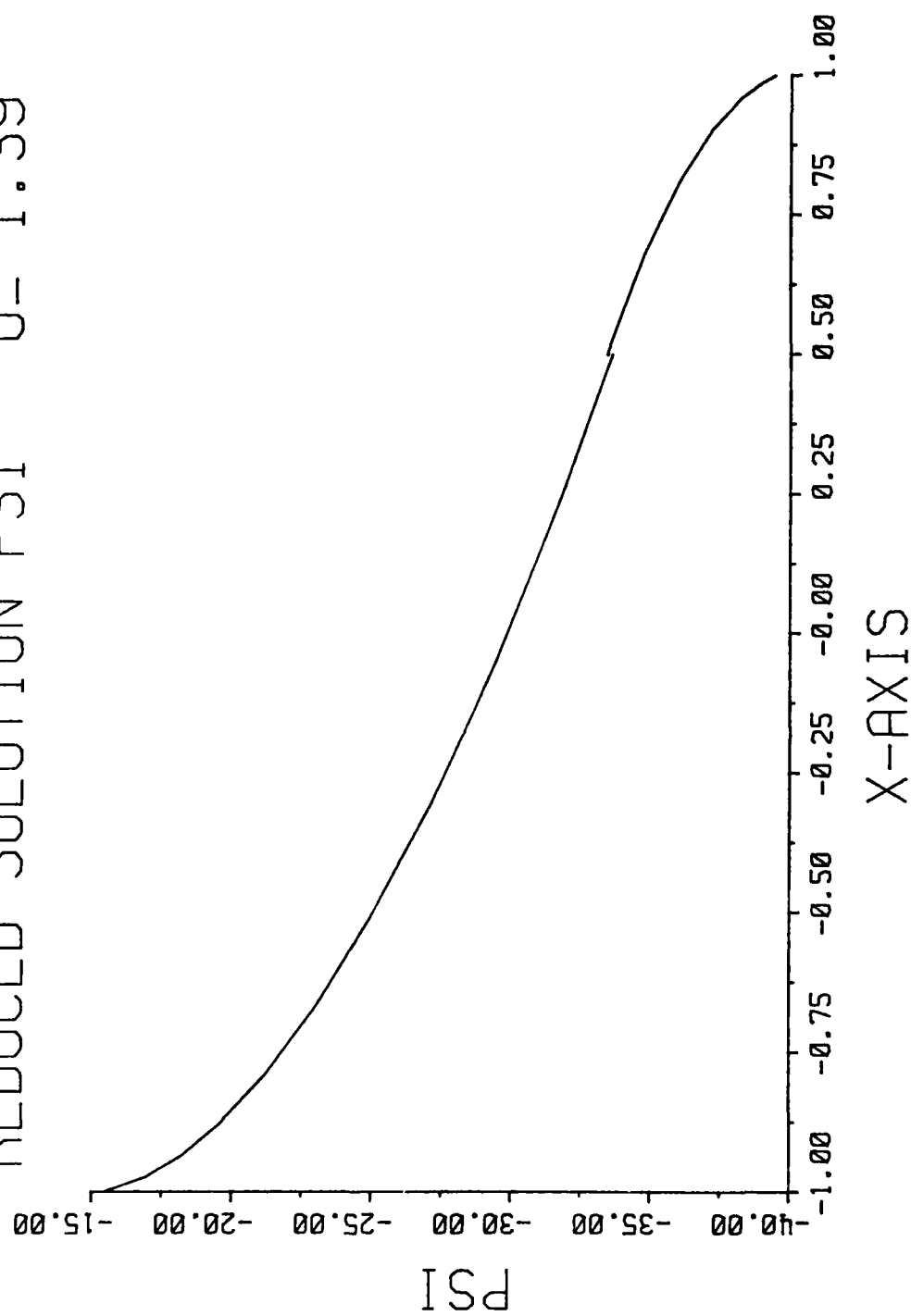
-48-

Figure 1

REDUCED SOLUTION D,N,P U= 1.39

Figure 2

REDUCED SOLUTION PSI   U= 1.39

X-AXIS

PSI

Figure 3

REDUCED SOLUTION JN, JP U= 1.39

X-AXIS

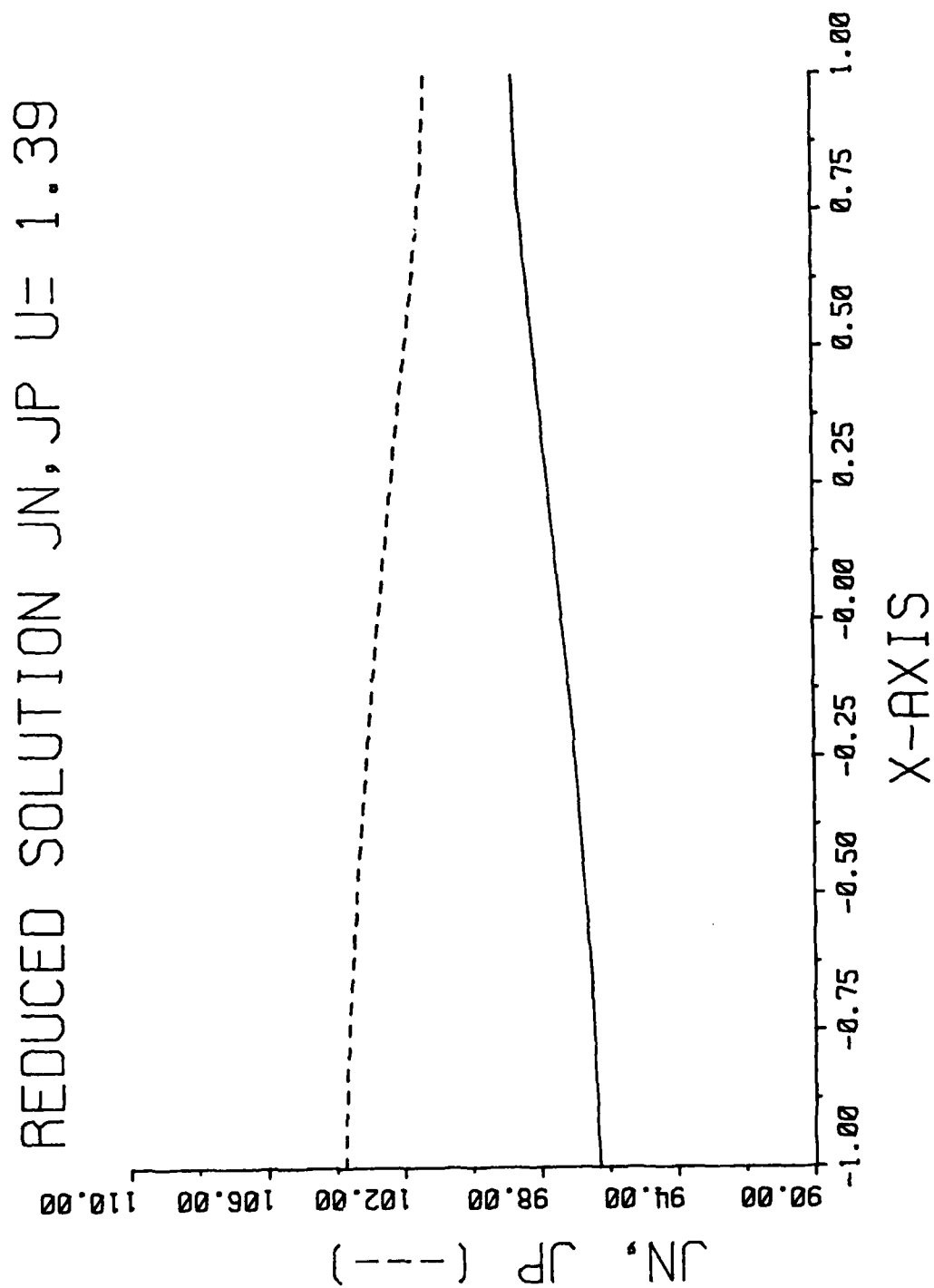Figure 4

REDUCED SOLUTION D,N,P U= 0.99
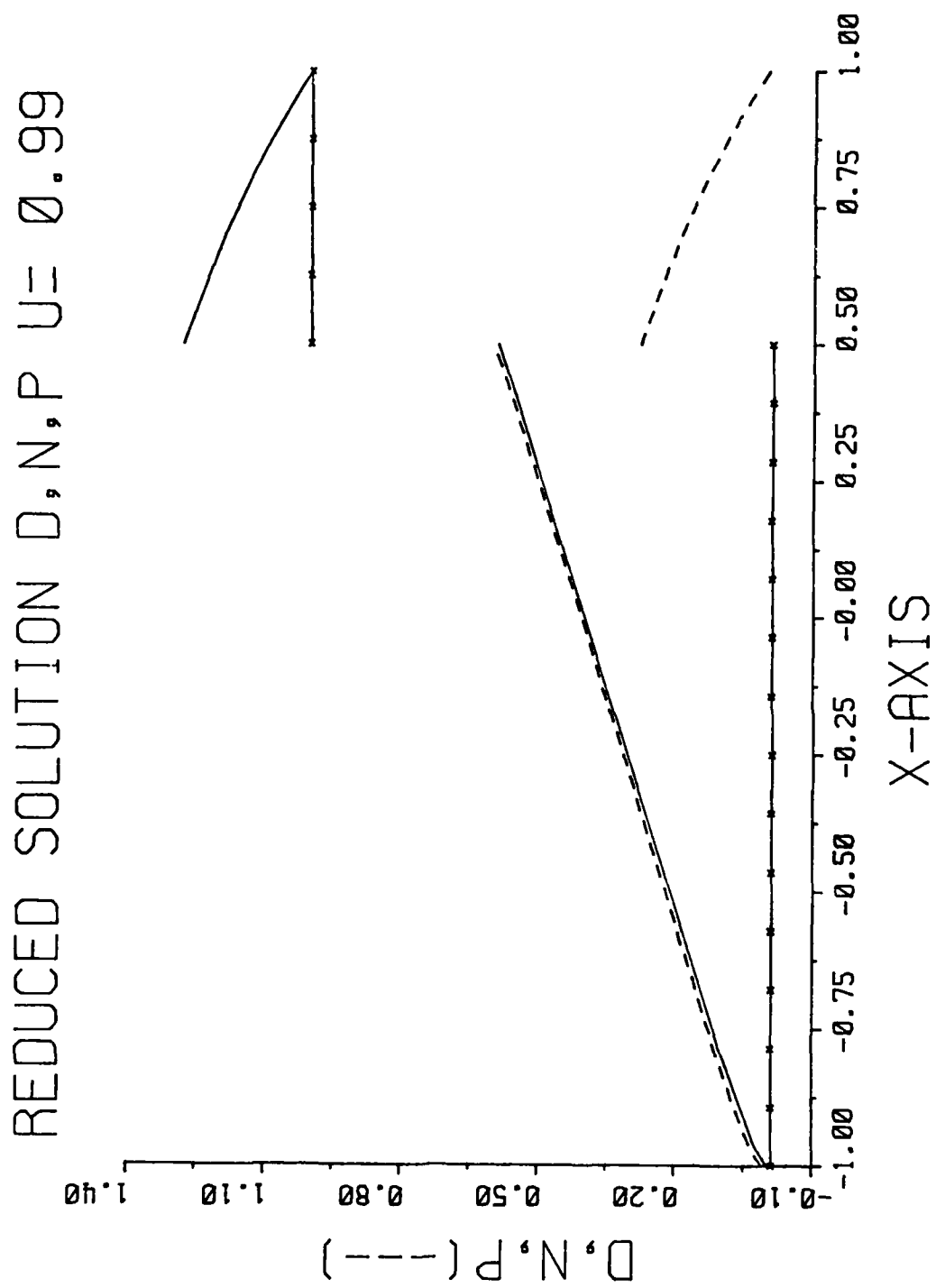

X-AXIS

D,N,P(---)
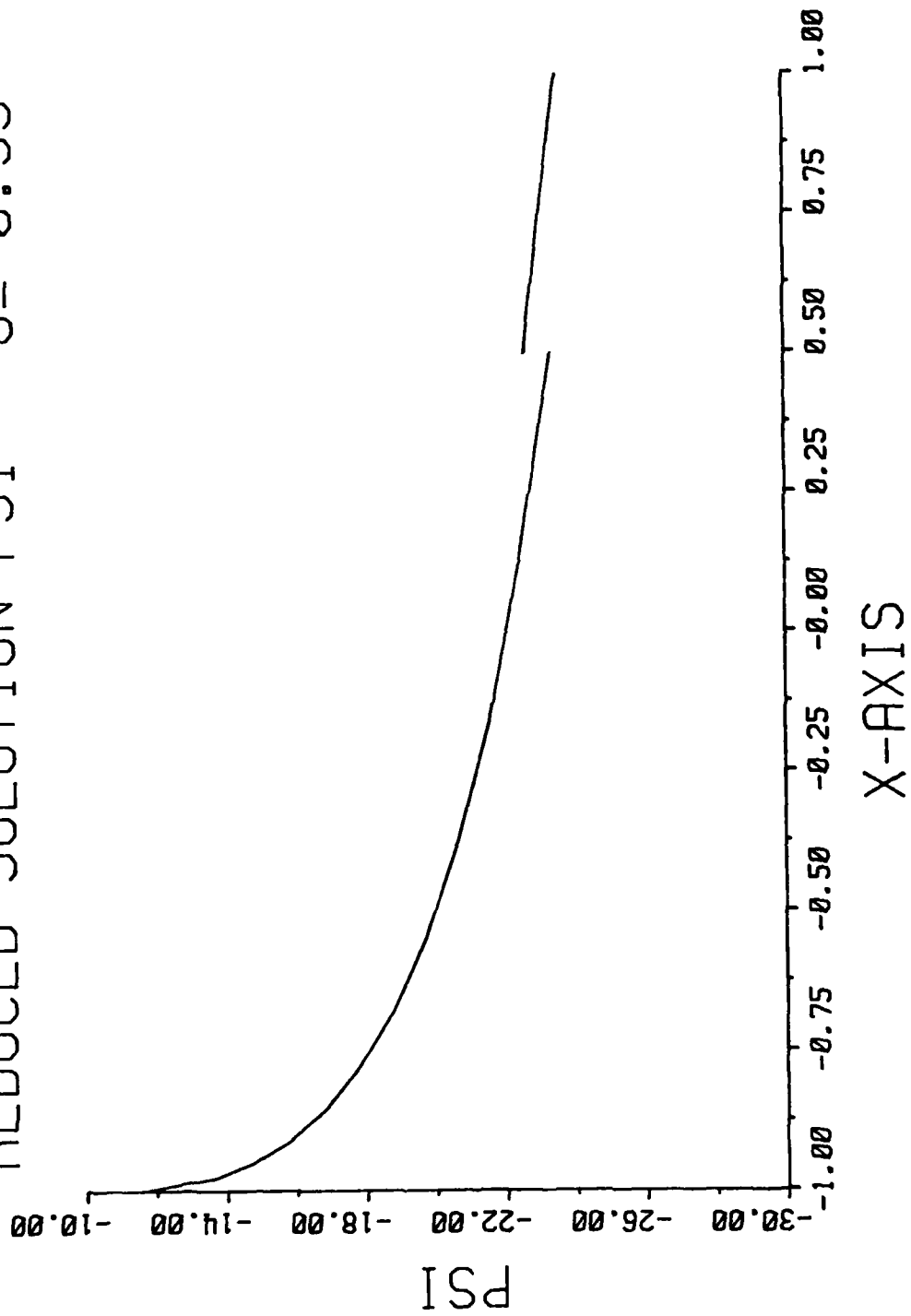
Figure 5

REDUCED SOLUTION PSI   U= 0.99

Figure 6

REDUCED SOLUTION JN, JP U= 0.99

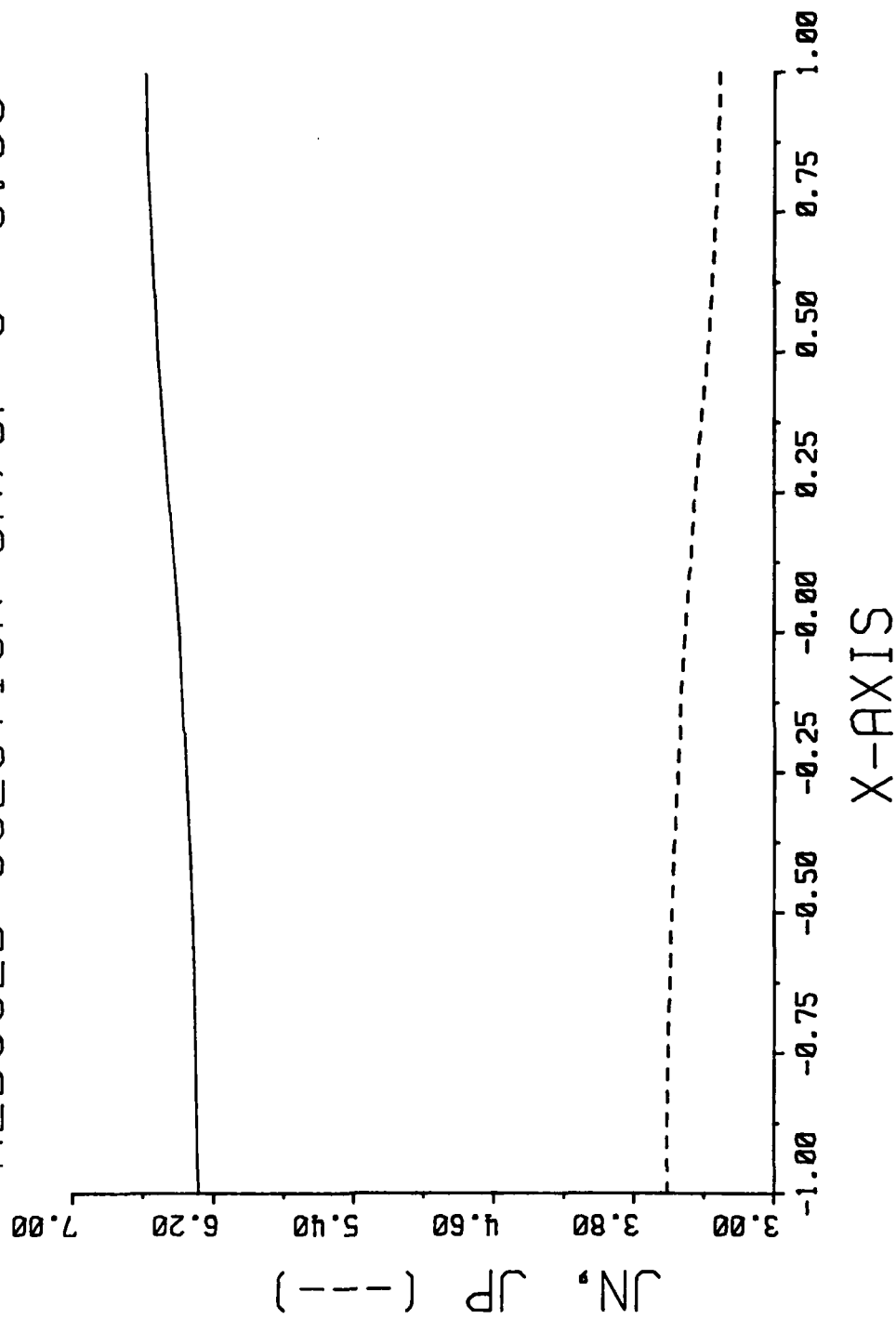X-AXIS

Figure 7

FULL SOLUTION D, N, P   U= 0.79

Figure 8
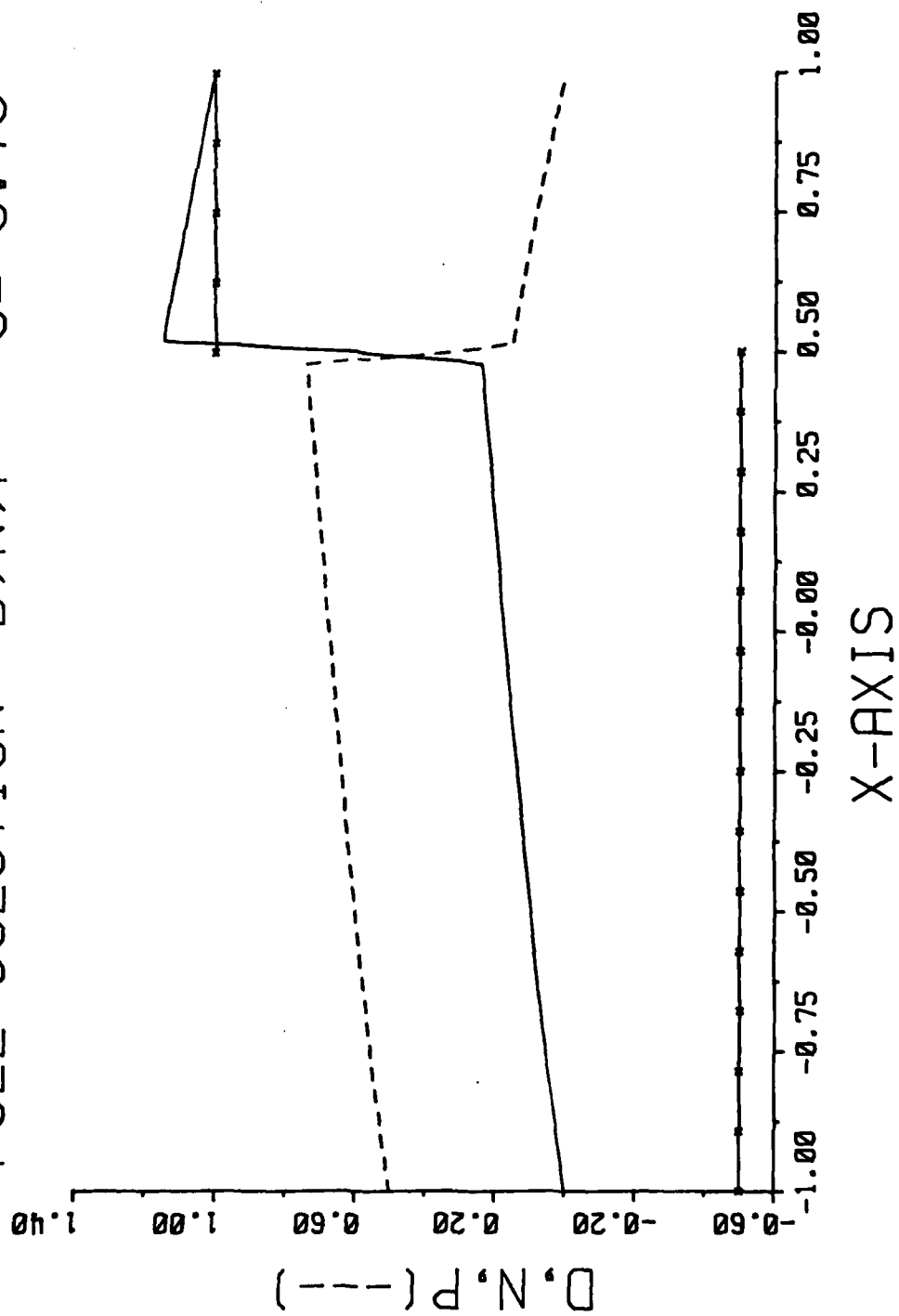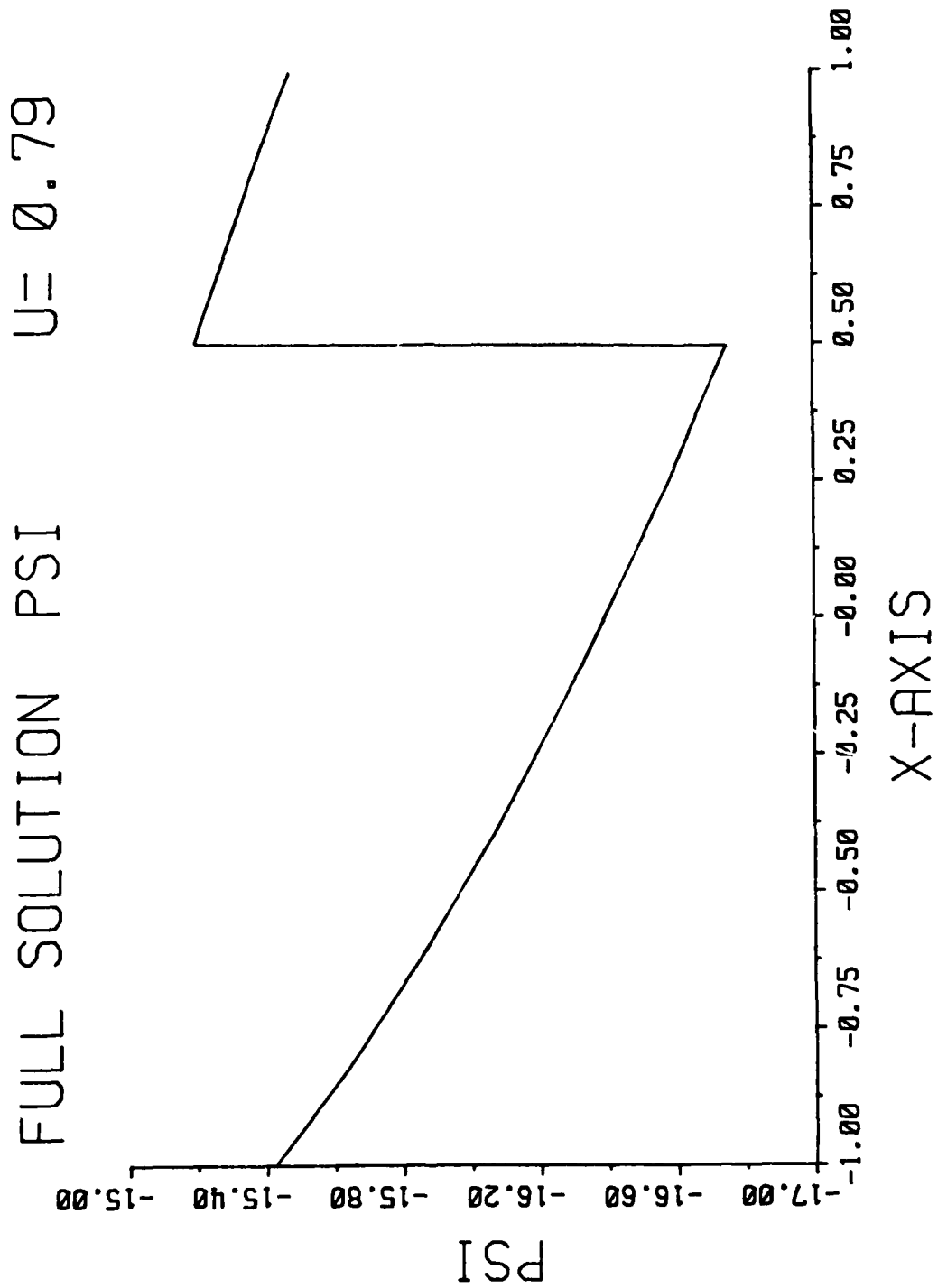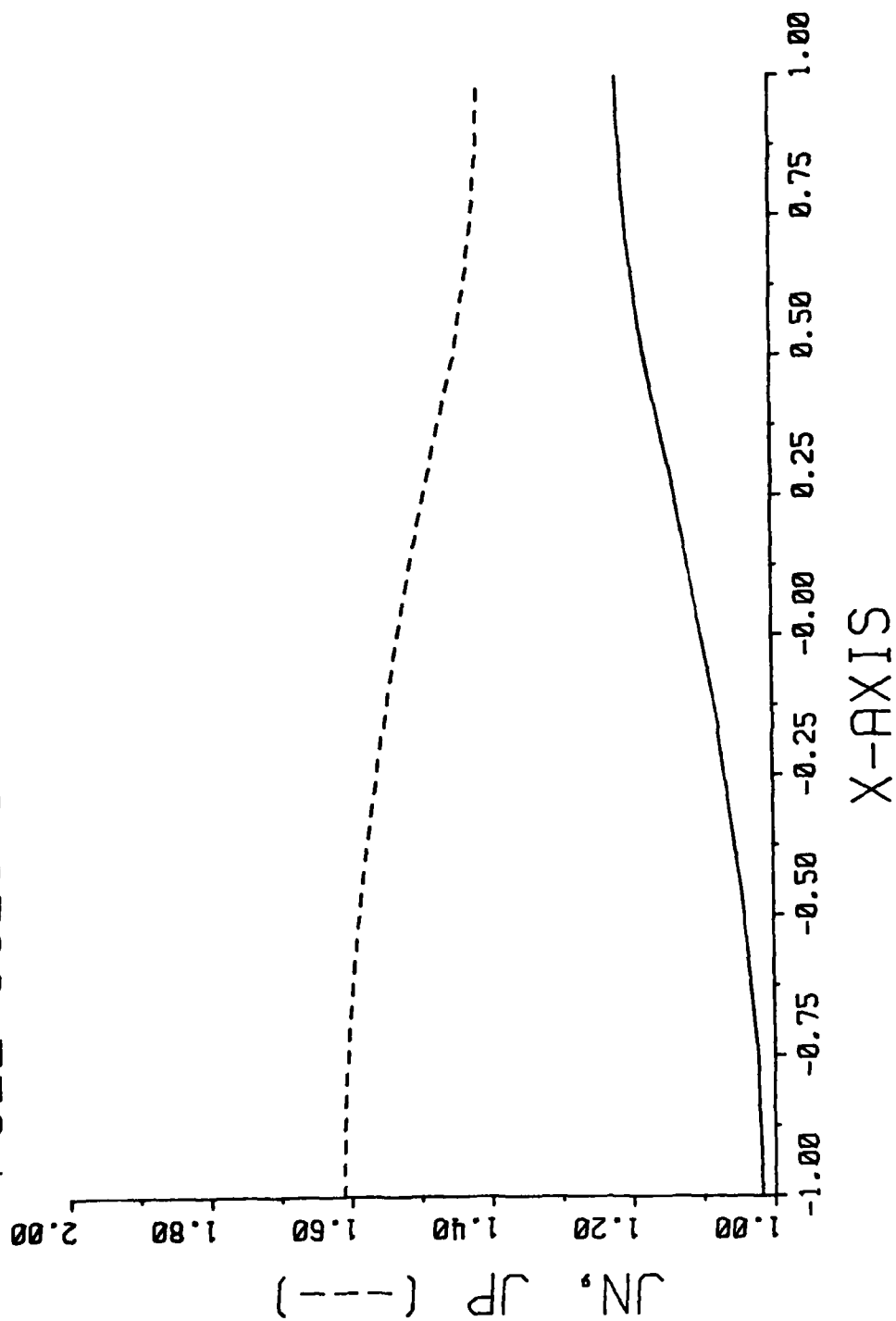
FULL SOLUTION PSI   U= 0.79

Figure 9

FULL SOLUTION JN, JP   U= 0.79

JN, JP (---)

X-AXIS

Figure 10

FULL SOLUTION D,N,P   U= 0.69

-58-
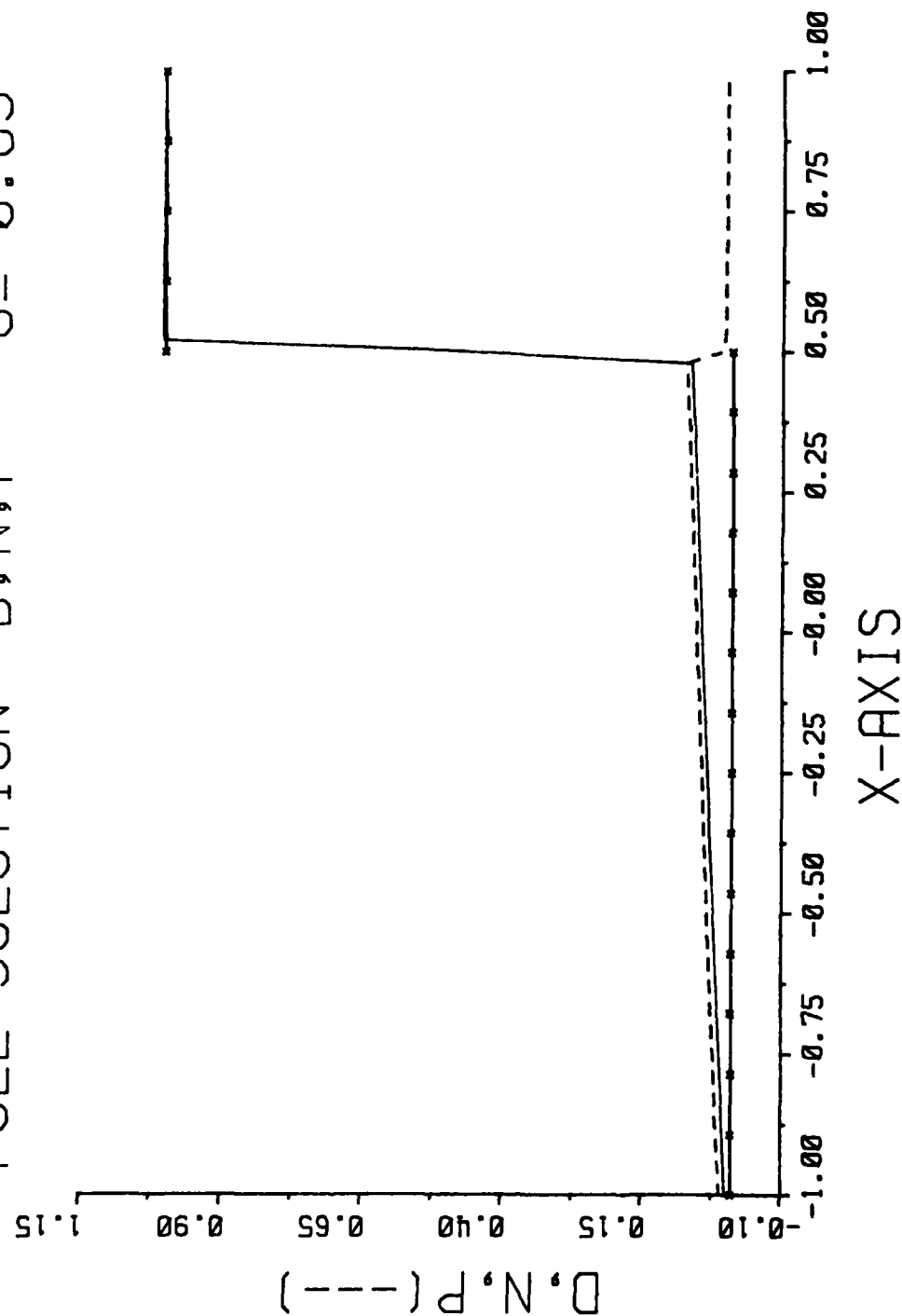
Figure 11

FULL SOLUTION PSI   U= 0.69

Figure 12

FULL SOLUTION JN, JP   U= 0.69

Figure 13

REDUCED SOLUTION D,N,P U= 0.79

D, N, P (---)

X-AXIS
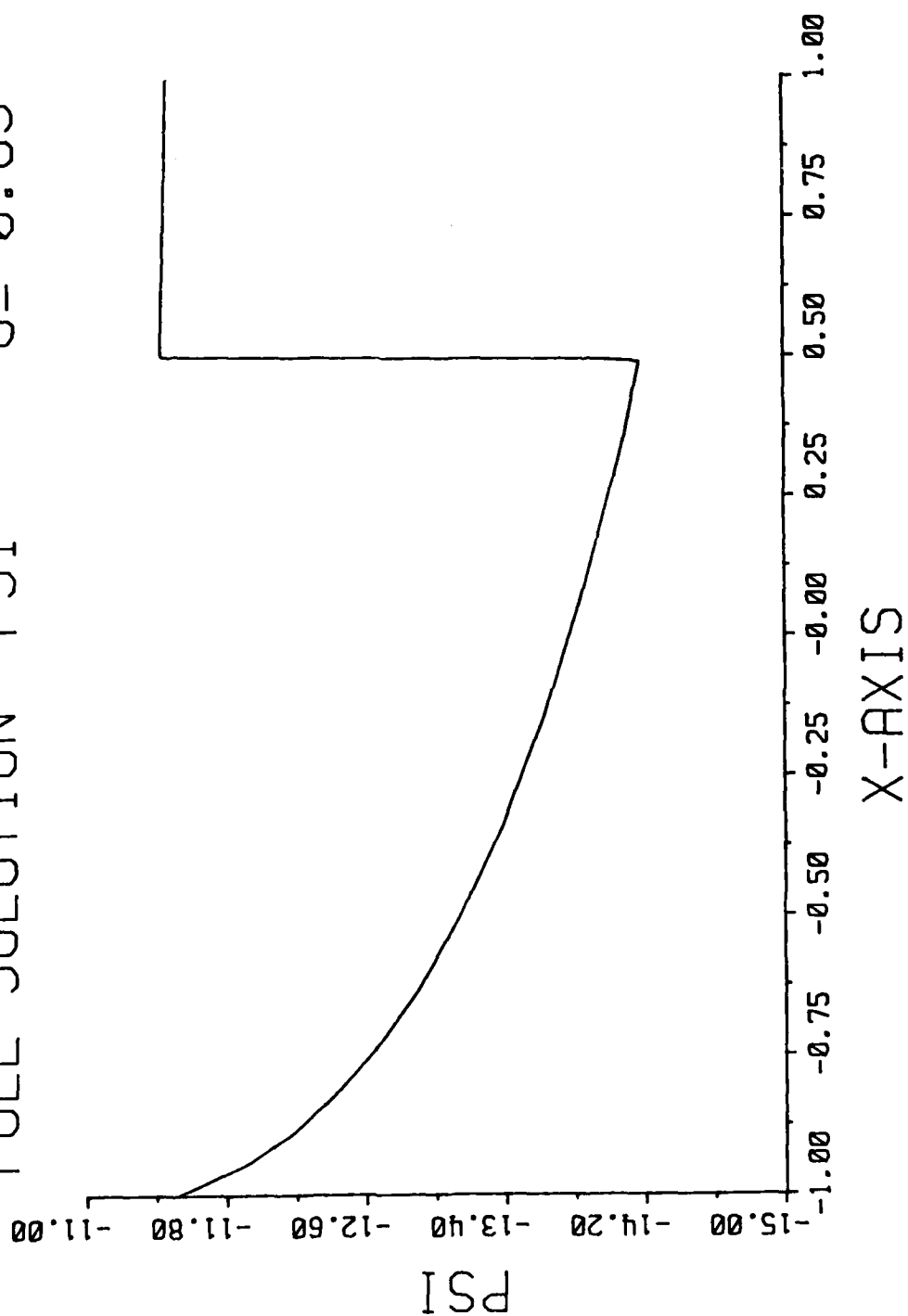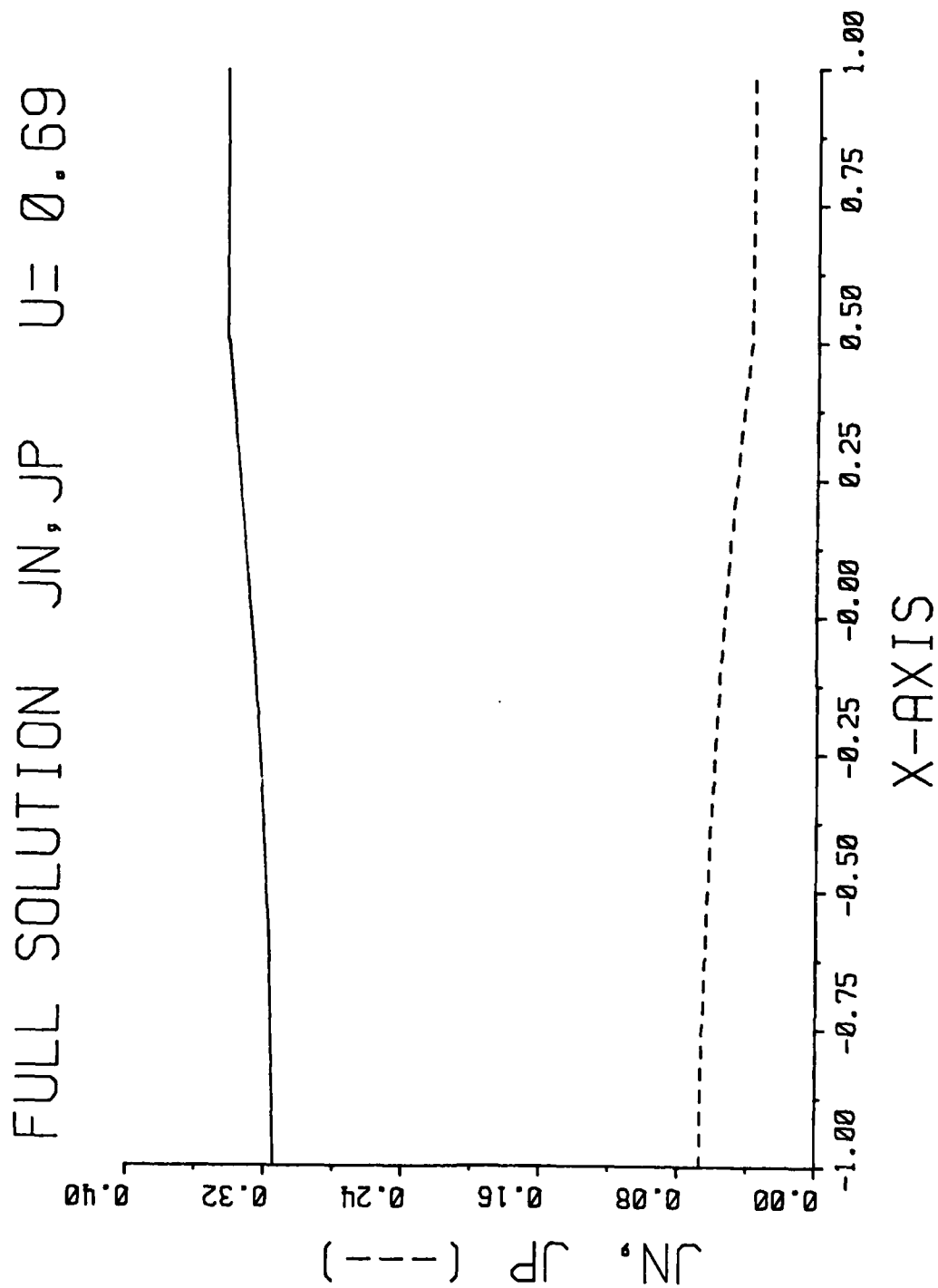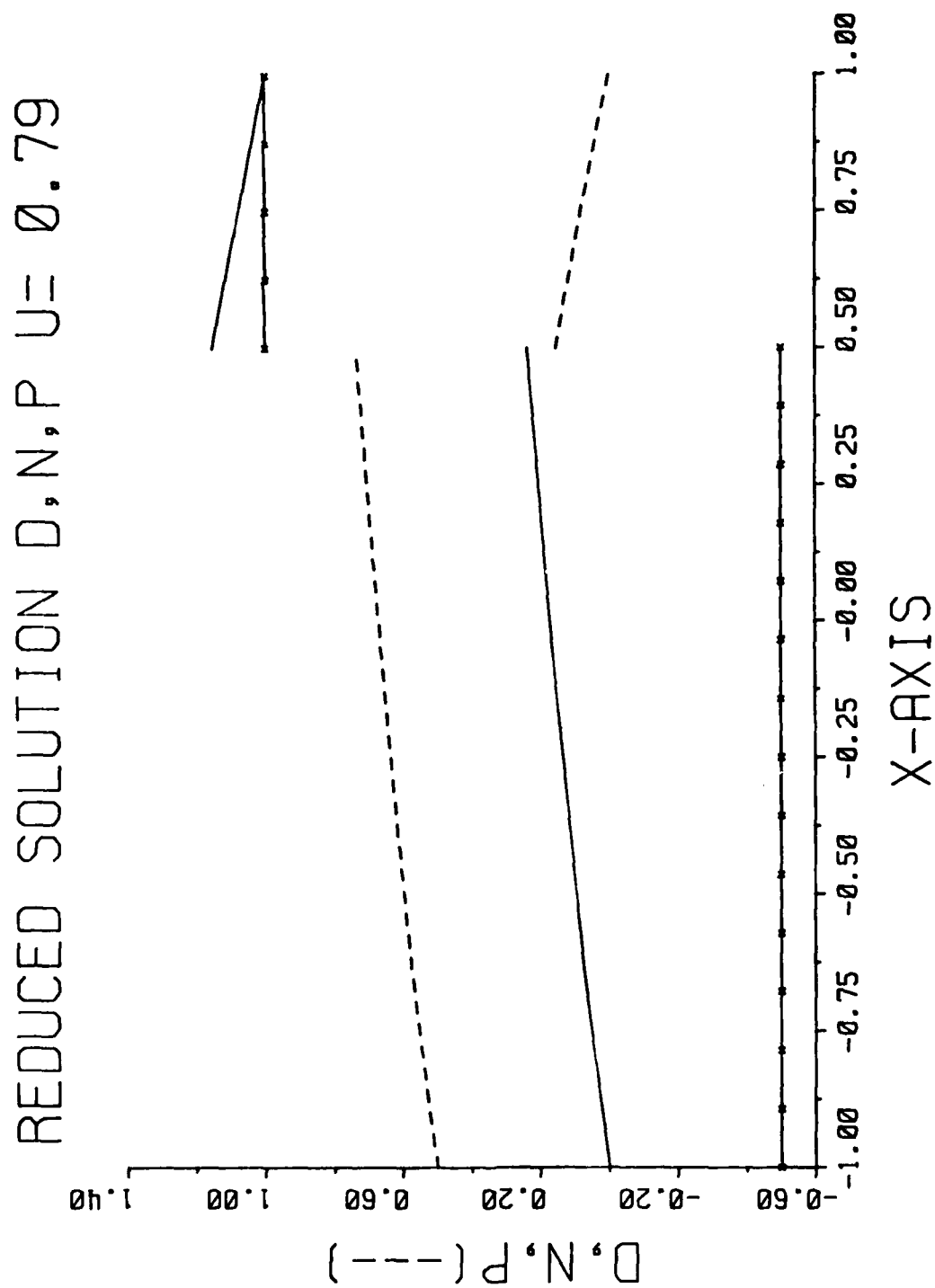
References

N. W. Ashcroft and N. D. Mermin (1976). "Solid State Physics", Saunders College

P. C. Fife (1973). 'Semilinear Elliptic Boundary Value Problems with Small Parameters',
Arch. Rat. Mech. Anal., Vol. 29, pp. 1-17.

E. Langer, S. Selberherr and H. Mader (1981). 'A Numerical Analysis of Bulk-Barrier
Diodes', to appear in Solid State Electronics.

P. A. Markowich, C. A. Ringhofer, S. Selberherr and E. Langer (1982). 'A Singularly
Perturbed Boundary Value Problem Modelling a Semiconductor Device', MRC-TSR #2388.

A. Schütz, S. Selberherr and H. W. Pötzl (1982). 'A Two Dimensional Model of the
Avalanche Effect in MOS Transitors', Solid State Electronics, Vol. 25, No. 3, pp.
177-183.

D. Smith (1980). 'On a Singularly Perturbed Boundary Value Problem Arising in the Physical
Theory of Semiconductors', Report, Institute für Mathematik und Informatik, TU
MUNCHEN.

R. A. Smith (1978). 'Semiconductors', 2nd Edition, Cambridge University Press.

M. N. Spijker (1972). 'Theorems for Nonlinear Finite Difference Methods', in
Numerische Lösung partieller Differential und Integrodifferential Gleichungen",
Lecture Notes in Mathematics, Vol. 267.

S. M. Sze (1969). "Physics of Semiconductor Devices', Wiley Interscience, New York-London-
Sydney-Toronto.

W. V. Roosbroeck (1950). 'Theory of Flow of Electrons and Holes in Germanium and
other Semiconductors', Bell Syst. Techn. J., Vol. 29, pp. 560-607.

A. B. Vasilev'a and V. F. Butuzov (1978). 'Singularly Perturbed Equations in the
Critical Case', Moscow State University. (Translated as MRC-TSR #2039.)

A. B. Vasilev'a and V. G. Stelmakh (1977). 'Singularly Disturbed Systems of the Theory of
Semiconductor Devices', USSR Comput. Math. Phys., Vol. 17, pp. 48-58.

PAM/CAR/EL/SS/jvs

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2527	**2. GOVT ACCESSION NO.** $AD\text{-}A130499$	**3. RECIPIENT'S CATALOG NUMBER**
4. TITLE (and Subtitle) An Asymptotic Analysis of Single-Junction Semiconductor Devices		**5. TYPE OF REPORT & PERIOD COVERED** Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Peter A. Markowich, C. A. Ringhofer, E. Langer and S. Selberherr		**8. CONTRACT OR GRANT NUMBER(s)** MCS-7927062, Mod. 2. DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		**10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS** Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below		**12. REPORT DATE** June 1983
		13. NUMBER OF PAGES 62
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		**15. SECURITY CLASS. (of this report)** UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

semiconductor devices, singularly perturbed ordinary differential equations, asymptotic expansions, internal layers.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we present an analysis of the fundamental one-dimensional semiconductor equations describing potential, carrier, and current density distributions in single-junction semiconductor devices when an external voltage is applied to the contacts. We reformulate the model equations by appropriate scaling as a singularly perturbed two point boundary value problem for a system of nonlinear ordinary differential equations. The right-hand side of the system has a jump discontinuity with respect to the independent variable (space-coordinate) representing the junction between differently doped sides of the

DD **FORM 1473**   EDITION OF 1 NOV 65 IS OBSOLETE   (continued)

ABSTRACT (continued)

device. The solution components are assumed to be continuous across this junction.

     We give an existence proof for the reduced problem (the singular perturbation parameter is set to zero). The discontinuity of the right-hand side of the system produces a discontinuity in the reduced potential and reduced carrier distributions. This creates an internal layer in the corresponding solution components of the singularly perturbed problem. The current distributions have no internal layer. We also derive the (internal) layer equations and give an existence proof. No boundary layers occur.

     We show that formal expansions actually represent (asymptotically) solutions of the singularly perturbed problem if the applied voltage is sufficiently small, and we investigate the ..pendence of the total current on the applied voltage. Numerical computations are reported.

END

FILMED

8-83

DTIC